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A note on Mahler's Theorem - II

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Abstract. For an irrational number α, it is well-known that the fractional part of $m\alpha$ is dense in the interval $(0, 1)$. This in particular implies that an integral multiple of α captures a given digit in base b at least once. In 1973, K. Mahler proved that some integral multiple of a given irrational number α captures a given block B of length n in base b infinitely often. However, the integer in Mahler's result is existential and lies in an interval $[1, 2b^{n+1}]$. In this short note, we explicitly find an interval I such that for all integers $X \in I$, the given block B of length n in base b occurs in the fractional part of $X\alpha$ with some frequency, under the assumptions of appearance of block of zeroes in α. Without these assumptions, finding the explicit values of integer X seems a difficult problem.

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1. Introduction

Let $b \geq 2$ be an integer. We say a non-zero real number α has *b-ary expansion*, if there exist $a_0 \in \mathbb{Z}$ and non-negative integers $a_1, a_2, \ldots, a_k, \ldots$ with $0 \leq a_k \leq b - 1$ such that

$$\alpha = a_0 + \frac{a_1}{b} + \frac{a_2}{b^2} + \cdots + \frac{a_k}{b^k} + \cdots . \tag{1}$$

and we write $\alpha = a_0.a_1a_2 \ldots a_n \ldots$, where a_i for all $1 \leq i$, are called *digits of α in base b*.

We denote, $B = a_1a_2 \ldots a_n$ to be *block of length n in base b*, formed by concatenating $n-$many a_k's, with $0 \leq a_k \leq b - 1$. As we shall be dealing with digits of $b-$ary of real numbers occurring after decimal point, we shall work with $\{\alpha\}$ which denotes the *fractional part of α*. Also with $[\alpha]$, we denote the *integral part of α*.

For a given real number α satisfying (1) and an integer $m \geq 1$; for a block B in base b, with $N(\alpha, B, m)$ we denote the number of times the block B occurs in the first m digits of the base b expansion of $\{\alpha\}$. Also, *frequency of a block B* in base b expansion of $\{\alpha\}$ is denoted by ν and is defined to be;

$$\lim_{m\to\infty} \frac{N(\alpha, B, m)}{m} = \nu,$$

It is less known about the digits or sequence of digits in the decimal expansion of a given irrational number, like $\sqrt{2}$ or π. It is comparatively easy to construct an irrational number such that in its decimal representation, certain digits or blocks of digits do not occur. The well-known theorems of Tchebychef, Kronecker, and Weyl imply that a given block of digits does occur at least once in the decimal representation of some integral multiple of a given irrational number, as the fractional part of the integral multiples of an irrational number is dense in the interval $(0, 1)$.

In 2008, B. Adamczewski and N. Rampersad [2] proved that *the binary expansion of an algebraic irrational contains infinitely many occurrences of* $7/3$ *powers. Also, every algebraic irrational contains either infinitely many occurrences of squares or infinitely many occurrences of one of the blocks* 010 *or* 02120 *in its ternary expansion.*

In another direction, an important result was proved by B. Adamczewski and Y. Bugeaud [1]. We denote the function $P(\alpha, b, n)$ to be the number of distinct blocks of length n in base b which occurs at least once in the base b expansion of α. Clearly, $1 \le P(\alpha, b, n) \le b^n$. Then they proved that *if* α *is an algebraic irrational number, then*

$$\lim_{n\to\infty} \frac{P(\alpha, b, n)}{n} = \infty.$$

In 1974, K. Mahler [6] proved that *if* α *is an irrational number, written in base* b *for some integer* $b \ge 2$*, and a given block* B *of length* n *in base* b*, there exists an integer* X *with* $1 \le X \le b^{2n+1}$ *such that the block* B *appears in the base* b *expansion of the fractional part of* $X\alpha$ *infinitely often.* There were improvements in the bounds on X (see [5] and [9]).

In 2017, N. K. Meher, K. Senthil Kumar and R. Thangadurai [7] proved that *if the block of* 0*'s of length* n *and followed by a non-zero digit occurs in the* b*-ary expansion of* α *with frequency* ν*, then there exists an integer* X *such that the given block* B *of length* m *(with* $1 \le m \le n$*) occurs in the fractional part of* $X\alpha$ *with the frequency at least* ν/b^{m+1}.

In [7], the integer X is only existential. One cannot pinpoint these are the integers X for which the above result holds. In the following theorem, we overcome this situation.

For a block $B = b_{k-1}b_{k-2}\dots b_0$ in base b, we associate an integer $L(B) = b_{k-1}b^{k-1} + b_{k-2}b^{k-2} + \dots + b_0$ and prove as follows.

Theorem 1. *Let* $b \ge 2$ *and* $1 \le r \le n$ *be integers. Let* $B = b_{r-1}b_{r-2}\dots b_1b_0$ *with* $b_i \ne 0$ *for some* $0 \le i \le r-1$ *be a given block of length* r *in the base* b*. Let* $\alpha \in (0, 1)$ *be an irrational number with* b*-ary expansion as,* $\alpha = \sum_{\ell=1}^{\infty} \frac{a_\ell}{b^\ell}$ *where* $a_\ell \in \{0, 1, \dots, b-1\}$ *such that for some natural number* h*, we have* $a_{h-n-r} = a_{h-n-r+1} = \dots = a_{h-1} = 0$ *and* $a_h = c_{n-1}, \dots, a_{h+n-1} = c_0$*. That is, the block* $A = \underbrace{00\dots0}_{n+r \text{ times}}C$ *where the block* $C = c_{n-1}c_{n-2}\dots c_0$ *with* $c_{n-1} \ne 0$*, occurs in* b*-ary expansion of* α*. Assume that* $0 < L(C)(L(C)+1) \le (L(C) - L(B))b^n$*. For any integer* X *lying in the interval* $\left[\frac{b^n L(B)}{L(C)}, \frac{b^n(L(B)+1)}{L(C)+1}\right)$*, if we write the fractional part of* $X\alpha$ *in base* b *as* $\{X\alpha\} = \sum_{\ell=1}^{\infty} \frac{z_\ell}{b^\ell}$*, then*

$$z_{h-n-r+1} = b_{r-1}, z_{h-n-r+2} = b_{r-2}, \dots, z_{h-n-1} = b_1, z_{h-n} = b_0.$$

Note. (1) When $n = 1$, we have $r = 1$. By the hypothesis in Theorem 1, we have $0 < a(a+1) \leq (a-b_0)b$ where $a = L(C) = C$ and $b_0 = L(B) = B$. This result proved by the second author [8]. The condition $a(a+1) \leq (a-b_0)b$ restricts the choices of b_0. However, in Theorem 1, if we choose $n = 2$ and $r = 1$, the condition $a(a+1) \leq (a-b_0)b^2$ with $a = L(C)$ and $C = 10$ allows to choose b_0 upto $b-1$ but we compramise the frequency of b_0 occuring in $\{X\alpha\}$.

(2) Though Theorem 1 is valid for any integer $b \geq 2$, for small values of n, we have restrictions to the block B due to the presence of the condition $L(C)(L(C)+1) \leq (L(C) - L(B))b^n$.

We have the following corollary.

COROLLARY 2

Let $b \geq 2$ and $1 \leq r \leq n$ be integers. Let $A = \underbrace{00\ldots0}_{n+r \text{ times}} c_{n-1}c_{n-2}\ldots c_0 = \underbrace{00\ldots0}_{n+r \text{ times}} C$ and $B = b_{r-1}b_{r-2}\ldots b_0$ be blocks of digits in base b such that $0 < L(C)(L(C)+1) \leq (L(C)-L(B))b^n$. Let $\alpha \in (0,1)$ be an irrational number with the block A occurring in b-ary expansion of α with frequency ν. Then the frequency of the block B occurring in $\{X\alpha\}$ is at least ν for all integers $X \in \left[\dfrac{b^nL(B)}{L(C)}, \dfrac{b^n(L(B)+1)}{L(C)+1}\right)$.

2. Proof of Theorems 1

Given that $\alpha \in (0,1)$ is an irrational number written in base b as $\alpha = 0.a_1a_2\cdots a_\ell\cdots$ and $B = b_{r-1}b_{r-2}\ldots b_1b_0$ is a given block in base b. Let the block $A = \underbrace{00\ldots0}_{n+r \text{ times}} c_{n-1}c_{n-2}\ldots c_0 = \underbrace{00\ldots0}_{n+r \text{ times}} C$ appear in α with the frequency ν. Assume that $h_1 < h_2 < \cdots < h_i < \cdots$ are the natural numbers (positions) such that $a_{h_i-n-r} = a_{h_i-n-r+1} = \cdots = 0 = a_{h_i-1}$ and $a_{h_i} = c_{n-1}, \ldots, a_{h_i+n-1} = c_0$ for each $i = 1, 2, \ldots$ and they determine the frequency ν. Let m be a natural number such that $a_{h_m-n-r} = a_{h_m-n-r+1} = \cdots = a_{h_m-1} = 0$ and $a_{h_m} = c_{n-1}\ldots a_{h_m+n-1} = c_0$. Let

$$s_m = \sum_{\ell=1}^{h_m-n-r-1} \frac{a_\ell}{b^\ell} \text{ and } t_m = \sum_{\ell=h_m}^{\infty} \frac{a_\ell}{b^{\ell-h_m+n}}.$$

Then $s_m \in \mathbb{Q}$ and t_m is irrational such that $t_m \in \left[\dfrac{L(C)}{b^n}, \dfrac{L(C)+1}{b^n}\right)$.

Let $I = \left[\dfrac{b^nL(B)}{L(C)}, \dfrac{b^n(L(B)+1)}{L(C)+1}\right)$ be the interval. Let $X \in I$ be any integer and $t \in \left[\dfrac{L(C)}{b^n}, \dfrac{L(C)+1}{b^n}\right)$ be any irrational number. Then

$$L(B) = \frac{b^nL(B)}{L(C)}\frac{L(C)}{b^n} \leq Xt < \frac{b^n(L(B)+1)}{L(C)+1}\frac{L(C)+1}{b^n} = L(B)+1$$

That is, any integer $X \in I$ satisfies $[Xt] = L(B)$ for all irrational numbers t lying in the interval $\left[\frac{L(C)}{b^n}, \frac{L(C)+1}{b^n}\right)$.

In particular, for t_m, we have $[Xt_m] = L(B)$.

By assumption

$$0 \le L(C)(L(C)+1) \le (L(C)-L(B))b^n \iff \frac{L(C)(L(C)+1)}{L(C)-L(B)} \le b^n$$
$$\iff \frac{b^n(L(B)+1)}{L(C)+1} - \frac{b^n L(B)}{L(C)} \ge 1.$$

This implies that the length of the interval I is ≥ 1 and hence there exists an integer in I. We fix one such integer $X \in I$. Note that

$$\alpha = s_m + \frac{t_m}{b^{h_m-n}} \implies X\alpha = Xs_m + \frac{Xt_m}{b^{h_m-n}} = Xs_m + \frac{[Xt_m]+\{Xt_m\}}{b^{h_m-n}} = Xs_m + \frac{L(B)}{b^{h_m-n}} + \frac{\{Xt_m\}}{b^{h_m-n}}.$$

Observe that $L(B) \le b^r - 1$ and $b^{n-1} \le L(C) \le b^n - 1$. Since X is an integer in I, we get

$$X < \frac{b^n(L(B)+1)}{L(C)+1} \le \frac{b^n(b^r)}{b^{n-1}+1} < b^{r+1}.$$

Therefore, we write $X = x_r b^r + \ldots + x_1 b + x_0$ in base b, where $x_i \le b-1$ for $i = 0, 1, \ldots, r$. Since the denominator of s_m is at most $b^{h_m-n-r-1}$, we see that Xs_m cannot give any carryover to the digits in $L(B)/b^{h_m-n} = [Xt_m]/b^{h_m-n}$. Note also that $\{X\{\alpha\}\} = \{X\alpha\}$. Therefore, we get

$$\{X\alpha\} = \sum_{\ell=1}^{h_m-n-r-1} \frac{d_\ell}{b^\ell} + \frac{L(B)}{b^{h_m-n}} + \frac{\{Xt_m\}}{b^{h_m-n}} \tag{2}$$

for some $d_j \in \{0, 1, \ldots, b-1\}$. Since $L(B) = b_{r-1}b^{r-1} + \cdots + b_1 b + b_0$, we see that

$$\frac{L(B)}{b^{h_m-n}} = \frac{b_{r-1}b^{r-1} + \cdots + b_1 b + b_0}{b^{h_m-n}} = \frac{b_{r-1}}{b^{h_m-n-r+1}} + \cdots + \frac{b_1}{b^{h_m-n-1}} + \frac{b_0}{b^{h_m-n}}.$$

For an integer $w \neq m$, we follow the above procedure to get

$$\{X\alpha\} = \sum_{\ell=1}^{h_w-n-r-1} \frac{e_\ell}{b^\ell} + \frac{L(B)}{b^{h_w-n}} + \frac{\{Xt_w\}}{b^{h_w-n}} \tag{3}$$

for some $d_j \in \{0, 1, \ldots, b-1\}$. Since $\{X\alpha\}$ is an irrational number, it has a unique expansion in base b. Therefore, by (1) and (3), we must have

$$\{X\alpha\} = \sum_{\ell=1}^{h_w-n-r-1} \frac{e_\ell}{b^\ell} + \frac{L(B)}{b^{h_w-n}} + \frac{\{Xt_w\}}{b^{h_w-n}} = \sum_{\ell=1}^{h_m-n-r-1} \frac{d_\ell}{b^\ell} + \frac{L(B)}{b^{h_m-n}} + \frac{\{Xt_m\}}{b^{h_m-n}}$$

and each respective digit is the same. Thus, we conclude that if the base b expansion of $\{X\alpha\}$ is

$$\{X\alpha\} = \sum_{\ell=1}^{\infty} \frac{z_\ell}{b^\ell}$$

then $z_{h_\ell-n-r+1} = b_{r-1}, z_{h_\ell-n-r+2} = b_{r-2} \ldots, z_{h_\ell-n} = b_0$ for each integer $\ell \ge 1$. Moreover, we see that the above procedure holds for any integer $Y \in I$. This proves the theorem. $\square$

3. Proof of Corollary 2

Let $\alpha = \sum_{\ell=1}^{\infty} \frac{a_\ell}{b^\ell}$ be the base b representation of the given irrational number α.
Since the block $A = \underbrace{00\ldots0}_{n+r \text{ times}} c_{n-1}c_{n-2}\ldots c_0 = \underbrace{00\ldots0}_{n+r \text{ times}} C$ occurs in α with a frequency ν, we have

$$\lim_{m\to\infty} \frac{N(\alpha, A, m)}{m} = \nu. \tag{4}$$

Let $X \in \left[\frac{b^n L(B)}{L(C)}, \frac{b^n(L(B)+1)}{L(C)+1}\right)$ be a given integer. Then, we need to prove

$$\lim_{m\to\infty} \frac{N(X\alpha, B, m)}{m} \geq \nu. \tag{5}$$

Let $\epsilon > 0$ be any given real number. Since the block A occurs in α with the frequency ν, by (4), there exists a constant M_0 such that

$$N(\alpha, A, m) \geq (\nu - \epsilon)m \tag{6}$$

for every $m \geq M_0$. Take any integer $m \geq M_0$. By (6), in the first m digits of the base b representation of α, the block $A = \underbrace{00\ldots0}_{n+r \text{ times}} C$ occurs in the positions $a_{h_i-n-r} = a_{h_i-n-r+1} = \cdots = a_{h_i-1} = 0$ and $a_{h_i} = c_{n-1}, a_{h_i+1} = c_{n-2}, \ldots, a_{h_i+n-1} = c_0$ for all $i = 1, 2, \ldots, \ell_m$ where $\ell_m = [(\nu - \epsilon)m]$, the integral part of $(\nu - \epsilon)m$. By Theorem 1, if we write $\{X\alpha\}$ in base b representation as $\{X\alpha\} = \sum_{\ell=1}^{\infty} \frac{d_\ell}{b^\ell}$, then we get

$$d_{h_i-n-r+1} = b_{r-1}, d_{h_i-n-r+2} = b_{r-2}, \ldots d_{h_i-n} = b_0 \text{ for all } i = 1, 2, \ldots, \ell_m.$$

Therefore, we get

$$N(X\alpha, B, m) \geq \ell_m = [(\nu - \epsilon)m] \geq (\nu - \epsilon)m - 1 = (\nu - \epsilon - 1/m)m.$$

This is true for all $m \geq M_0$. Therefore, we get

$$\lim_{m\to\infty} \frac{N(X\alpha, B, m)}{m} \geq \lim_{m\to\infty} (\nu - \epsilon - 1/m) = \nu - \epsilon$$

holds for any $\epsilon > 0$. Therefore (5) holds true and proves the corollary. □

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