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par Florian Luca et Ravindranathan Thangadurai

Résumé. Soit $n$ un nombre entier positif et $p(n)$ le plus grand nombre premier $p \leq n$. On considère la suite finie décroissante définie récursivement par $n_1 = n$, $n_{i+1} = n_i - p(n_i)$ et dont le dernier terme, $n_r$, est soit premier soit égal à 1. On note $R(n) = r$ la longueur de cette suite. Nous obtenons des majorations pour $R(n)$ ainsi qu’une estimation du nombre d’éléments de l’ensemble des $n \leq x$ en lesquels $R(n)$ prend une valeur donnée $k$.

Abstract. For every positive integer $n$ let $p(n)$ be the largest prime number $p \leq n$. Given a positive integer $n = n_1$, we study the positive integer $r = R(n)$ such that if we define recursively $n_{i+1} = n_i - p(n_i)$ for $i \geq 1$, then $n_r$ is a prime or 1. We obtain upper bounds for $R(n)$ as well as an estimate for the set of $n$ whose $R(n)$ takes on a fixed value $k$.

1. Introduction

Let $n > 1$ be an integer. Let $p(n)$ be the largest prime factor of $n$. Let $n_2 = n_1 - p(n_1)$. If $n_2 > 1$, let $n_3 = n_2 - p(n_2)$, and, recursively, if $n_k > 1$, we put $n_{k+1} = n_k - p(n_k)$. Note that if $n_k$ is prime, then $n_{k+1} = 0$. We put $R(n)$ for the positive integer $k$ such that $n_k$ is prime or 1. Hence, we obtain a representation of $n$ of the form

\[ n = p_1 + p_2 + \cdots + p_r, \]

with $r = R(n)$, where $p_1 > p_2 > \cdots > p_r$ are primes except for the last one which might be 1.

The above representation of $n$ was first considered by Pillai in [6] who obtained a number of interesting results concerning the function $R(n)$. Here, we extend some of Pillai’s results on this function.

Since by Bertrand’s postulate the interval $[x, 2x)$ contains a prime number for all $x \geq 1$, it follows that if $n_k > 1$, then $n_{k+1} \leq n_k/2$. This immediately implies that $R(n) = O(\log n)$. Pillai proved that the better estimate $R(n) = o(\log n)$ holds as $n \to \infty$. He also showed, under the Riemann Hypothesis, that the inequality $R(n) < 2\log \log n$ holds whenever $n > n_0$.

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Here, we remove the conditional assumption on the Riemann Hypothesis from Pillai’s result and prove the following theorem.

**Theorem 1.1.** The estimate

\[ R(n) \ll \log \log n \]

holds for all positive integers \( n \geq 3 \).

Pillai also showed that

\[ (1.2) \quad \limsup_{n \to \infty} R(n) = \infty. \]

Our next result is slightly stronger than estimate (1.2) above. In what follows, we put \( \log_k x \) for the function defined inductively as \( \log_1 x = \log x \) and \( \log_k x = \max\{1, \log(\log_{k-1} x)\} \) for \( k > 1 \). When \( k = 1 \), we omit the subscript. Note that if \( x \) is large, then \( \log_k x \) coincides with the \( k \)th fold composition of the natural logarithm function evaluated in \( x \).

**Theorem 1.2.** Let \( k \geq 1 \) be any fixed integer. Then the estimate

\[ \# \{ n \leq x : R(n) = k \} \asymp x \frac{1}{\log_k x} \]

holds.

Theorem 1.2 shows that for any fixed \( k \), the asymptotic density of the set of \( n \) with \( R(n) \leq k \) is zero. This shows not only that estimate (1.2) holds, but that \( R(n) \to \infty \) holds on a set of \( n \) of asymptotic density 1.

Pillai also conjectured that perhaps the inequality \( R(n) \gg \log \log n \) holds for infinitely many \( n \). We believe this conjecture to be false. Indeed, a widely believed conjecture of Cramér [2] from 1936, asserts that if \( x > x_0 \), then the interval \( [x, x + (\log x)^2] \) contains a prime number. If true, this implies that if \( n_k > x_0 \), then \( n_{k+1} < (\log n_k)^2 \). Let \( f(n) \) be the function which associates to each integer \( n > x_0 \) the minimal number of iterations of the function \( x \mapsto (\log x)^2 \) required to take \( n \) just below \( x_0 \). Then Cramér’s conjecture implies that \( R(n) \leq f(n) + O(1) \), where the constant implied in \( O(1) \) can be taken to be \( \max\{R(n) : n \leq x_0\} \). Let us take a look at these iterations. Assume that \( n \) is large. We then have \( n_1 = n, n_2 \leq (\log n)^2, n_3 \leq (\log n_2)^2 \leq 2(\log(2 \log n))^2 < 8(\log \log n)^2 \). Inductively, one shows that if \( k \) is fixed and \( n \) is sufficiently large with respect to \( k \), then the inequality \( n_k < 8(\log_k n)^2 \) holds. Since \( k \) is arbitrary, we conclude that \( f(n) = o(\log_k n) \) holds with any fixed \( k \geq 1 \) as \( n \to \infty \), so, in particular, the inequality \( f(n) \gg \log \log n \) cannot hold for infinitely many positive integers \( n \). Let us observe that the weaker assumption that the interval \( [x, x + \exp((\log x)^{1/2})] \) contains a prime for all \( x > x_0 \) will easily lead to the conclusion that \( R(n) = O(\log_3 n) \). Indeed, in this case we have \( \log n_{k+1} \leq (\log n_k)^{1/2} \), whenever \( n_k > x_0 \). In particular, \( \log n_{k+1} \leq (\log n)^{1/2k} \), whenever \( n_{k+1} > x_0 \). This implies
easily that for some \( k \) of size at most \((\log \log \log n)/\log 2 + O(1)\) we have \( n_{k+1} < x_0 \), so that \( R(n) = O(\log_3 n) \).

Pillai also looked at the sequence of local maxima (in modern terms also called champions) for the function \( R(n) \). Recall that \( n \) is called a champion if \( R(m) < R(n) \) holds for all \( m < n \). Let \( \{t_k\}_{k \geq 1} \) be the sequence of champions. Pillai showed that \( t_1 = 1 \) and that the recurrence \( t_{k+1} = p(t_{k+1}) + t_k \) holds for all \( k \geq 1 \). Furthermore, \( t_k \) and \( t_{k+1} \) have different parities for all \( k \geq 1 \). He also showed that \( \{t_k\}_{k \geq 1} \) grows very fast, namely that for each positive constant \( A \) one has \( t_{k+1} \gg_A t_k (\log t_k)^A \). He also calculated the first 4 values of the sequence \( \{t_k\}_{k \geq 1} \) obtaining

\[
 t_1 = 1, \quad t_2 = 4 = 3 + 1, \quad t_3 = 27 = 23 + 4, \quad t_4 = 1354 = 1327 + 27.
\]

He mentioned (seventy years ago!) that it is perhaps possible to compute \( t_5 \) but not \( t_6 \). Consulting Thomas Nicely’s [5] tables of prime gaps, we get

\[
 t_5 = 401429925999155061 = 401429925999153707 + 1354
\]

and Cramer’s conjecture implies that \( t_6 > \exp(4 \cdot 10^8) \), so indeed it is perhaps not possible to compute \( t_6 \).

2. Proof of Theorem 1.1

For the proof of the fact that \( R(n) < 2 \log \log n \) for \( n > n_0 \) under the Riemann Hypothesis, Pillai used the known consequence of the Riemann Hypothesis that for each \( \delta > 0 \), there is some \( x_\delta \) such that when \( x > x_\delta \), the interval \([x, x + x^{1/2+\delta}]\) contains a prime number.

In the same year as Pillai’s paper [6] appeared, Hoheisel proved his famous theorem about Prime Number Gaps.

**Theorem 2.1** ([4]). There exist absolute constants \( \theta \in (0,1) \) and \( N_0 \) such that for every integer \( n \geq N_0 \), the interval \([n - n^\theta, n]\) contains a prime number.

The best known \( \theta = 0.525 \) is due to Baker, Harman and Pinz [1]. The proof of Theorem 1.1 follows easily from Pillai’s arguments by replacing the prime number gaps guaranteed by the Riemann Hypothesis with Hoheisel’s result.\(^1\)

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\(^1\)It seems likely that Pillai was not aware of Hoheisel’s paper [4].
Let \( n_1 \geq N_0 \). By Theorem 2.1, \( p(n_1) > n - n^\theta \). Thus, the chain of inequalities

\[
\begin{align*}
n_2 &= n_1 - p(n_1) < n_1 - n_1 + n_1^\theta = n_1^\theta; \\
n_3 &= n_2 - p(n_2) < n_1^\theta < n_1^\theta^2; \\
n_4 &< n_1^\theta^3; \\
\ldots \ldots \\
n_{\ell+1} &< n_1^\theta^\ell
\end{align*}
\]

holds as long as \( n_\ell \geq N_0 \). We now let \( \ell \) be that integer such that \( n_{\ell+2} < N_0 \leq n_{\ell+1} \). We then have

\[ n_1^\theta^\ell \geq N_0, \]

therefore

\[ \theta^\ell \log n_1 \geq \log N_0, \]

which implies that

\[ \ell \log \theta + \log \log n_1 \geq \log \log N_0. \]

Hence,

\[ \log \log n_1 \geq \ell \log (1/\theta), \]

which in light of the fact that \( \theta \in (0, 1) \) gives

\[ \ell \leq \frac{\log \log n_1}{\log (1/\theta)}. \]

Put \( b = \max_{1 \leq m \leq N_0} \{ R(m) \} \). Trivially, \( b \leq \pi(N_0) \). Thus,

\[ R(n_1) \leq \ell + 1 + b < \frac{\log \log n_1}{\log (1/\theta)} + 1 + b \ll \log \log n_1, \]

which is the desired inequality.

3. Proof of Theorem 1.2

For every prime number \( p \) we put \( p' \) for the next prime following \( p \). The following result is certainly well-known but we shall supply a short proof of it.

**Lemma 3.1.** For \( 2 \leq y \leq \log x \), put

\[ \mathcal{P}(x, y) = \left\{ p \leq x : p' - p \not\in [y^{-1}(\log x), y(\log x)] \right\}. \]

Then,

\[ \#\mathcal{P}(x, y) \ll \frac{\pi(x)}{y}. \]
Proof. We first look at the primes \( p \leq x \) which are in \( \mathcal{P}(x,y) \) and \( p' - p > y \log x \). The interval \([1,x]\) is contained in the union of the subintervals \([(i-1)y \log x, iy(\log x)] \) for \( i = 1, 2, \ldots, \lfloor x/(y \log x) \rfloor + 1 \). Since \( p' - p > y(\log x) \), each one of the above intervals can contain at most one such prime \( p \). Thus, the number of such primes \( p \) does not exceed

\[
\#\{p \leq x : p' - p > y(\log x)\} \leq \lfloor x/(y \log x) \rfloor + 1 \leq 2x/(y \log x)
\]

(3.2)

We next look at the primes \( p \leq x \) which are in \( \mathcal{P}(x,y) \) and \( p' - p = h < z = y^{-1}(\log x) \). We fix \( h \) and look at the set of primes \( p \leq x \) such that \( p + h \) is also prime. We write \( \mathcal{A}_h(x) \) for this set. By Brun’s sieve (see, for example, [3, Theorem 5.7]), we have

\[
\#\mathcal{A}_h(x) \ll \frac{x}{(\log x)^2} \frac{h}{\phi(h)}.
\]

Summing up over all the acceptable values of \( h \leq z \), we get that

\[
\#\{p \leq x : p' - p < z\} \leq \sum_{1 \leq h \leq z} \#\mathcal{A}_h \leq \frac{x}{(\log x)^2} \sum_{1 \leq h \leq z} \frac{h}{\phi(h)} \ll \frac{xz}{(\log x)^2} \ll \frac{\pi(x)}{y}.
\]

(3.3)

In the above estimates, we used the known fact that the estimate

\[
\sum_{1 \leq h \leq t} \frac{h}{\phi(h)} \ll t
\]

holds for all \( t \geq 1 \) (see, for example, [7]). The desired conclusion follows now immediately from estimates (3.2) and (3.3). \( \square \)

Proof of Theorem 1.2. We put \( \mathcal{R}_k = \{n : R(n) = k\} \) and \( \mathcal{R}_k(x) = \mathcal{R}_k \cap [1,x] \). We prove the theorem by induction on \( k \) having as a base the case \( k = 1 \) for which the assertion is immediate by the Prime Number Theorem.

Assume that \( k \geq 2 \). We first deal with the upper bound on \( \#\mathcal{R}_k(x) \). We have, by the induction hypothesis,

\[
\#\mathcal{R}_k(x) = \#\{n = p + m \leq x : R(m) = k - 1, \ p \leq n < p'\}
\]

\[
= \sum_{p \leq x} \#\{m \leq p' - p : R(m) = k - 1\}
\]

\[
\leq \sum_{p \leq x} \#\mathcal{R}_{k-1}(p' - p) \ll_k \sum_{p \leq x} \frac{(p' - p)}{\log_k(p' - p)}.
\]

(3.4)
We split the last sum above at \( z = (\log x)^{1/3} \). If \( p' - p > z \), then \( \log_{k-1}(p' - p) \gg_k \log_k x \), therefore

\[
\sum_{\substack{p \leq x \\ p' - p > z}} \frac{(p' - p)}{\log_{k-1}(p' - p)} \ll_k \frac{1}{\log_k x} \sum_{p \leq x} (p' - p) \ll \frac{x}{\log_k x},
\]

where for the last inequality above we used the fact that the intervals \([p, p']\) for \( p \leq x \) are disjoint and their union is contained in \([1, 2x]\) by the Bertrand postulate. For the range \( p' - p \leq z \), we proceed as in the proof of Lemma 3.1 by first fixing \( h \leq z \) and looking at the primes \( p \in A_h(x) \). The proof of Lemma 3.1 shows that

\[
\sum_{p \in A_h(x)} \frac{(p' - p)}{\log_{k-1}(p' - p)} \ll \sum_{p \in A_h(x)} h \leq h \# A_h \ll \frac{x}{\log x} \frac{h^2}{\phi(h)} \ll \frac{xz}{\log x} \frac{h}{\phi(h)},
\]

therefore

\[
\sum_{\substack{p \leq x \\ p' - p \leq z}} \frac{(p' - p)}{\log_{k-1}(p' - p)} \ll \frac{xz}{\log x} \sum_{h \leq z} \frac{h}{\phi(h)} \ll \frac{xz^2}{\log x} = \frac{x}{z} \ll \frac{x}{\log_k x}.
\]

Estimates (3.4), (3.5) and (3.6) imply the desired upper bound on \( \# R_k(x) \).

We now turn our attention on the lower bound for \( \# R_k(x) \). We proceed again by induction on \( k \geq 1 \). Let \( c_1 > 0 \) be the constant implied in inequality (3.1) and let \( y = 2c_1 \). Then \( \# P(x, y) \leq \pi(x)/2 \). Let \( p \leq x \) be a prime not in \( \# P(x, y) \) and \( m \in R_{k-1}((\log x)/y) \). Put \( n = m + p \). Then \( n = m + p < (\log x)/y + p < p' \), therefore \( p = p(n) \). Thus, \( R(n) = 1 + R(m) = k \).

The number of pairs \((p, m)\) with the above properties is

\[
\geq (\pi(x) - \# P(x, y)) \# R_{k-1}((\log x)/y) \gg_k \frac{\pi(x) \log x}{\log_{k-1}((\log x)/y)} \gg_k \frac{x}{\log_k x}.
\]

Each such pair \((p, m)\) leads to a value of \( n \leq x + (\log x)/y \leq 2x \). Furthermore, distinct pairs \((p, m)\) lead to distinct values of \( n \), for if \( p + m = p_1 + m_1 \) for some \((p, m) \neq (p_1, m_1)\) then, assuming say that \( p_1 > p \), we get

\[
p' - p \leq p_1 - p = m - m_1 < m < (\log x)/y,
\]

which is impossible. Hence, \( p_1 = p \) and since \( p + m = p_1 + m_1 \), we also get \( m = m_1 \), which is impossible since the pairs \((p, m)\) and \((p_1, m_1)\) were distinct. Thus, we showed that \( \# R_k(2x) \gg_k x/\log_k x \), which implies the desired lower bound.
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