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A NOTE ON THE DISTRIBUTION OF VALUES OF AN ARITHMETICAL FUNCTION

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In this paper, we investigate the function

$$V(\lambda) \stackrel{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N} \sum_{n \le N,\, \delta_k(n) < n\lambda} 1 \qquad (0 \le \lambda \le 1)$$

where the arithmetical function $\delta_k(n)$ is defined by

$$\delta_k(n) = \max \{d : d \mid n,\ (d, k) = 1\}.$$

Apart from some standard techniques in analytic number theory, our method mainly consists of the application of a theorem of Delange[6] and an estimation of the number of integer lattice points in tetrahedrons.

1. INTRODUCTION

Let $k = p_1, \ldots, p_r$ be a squarefree natural number. The arithmetical function $\delta_k(n)$ is defined by $\delta_k(n) = \max \{d : d \mid n,\ (d, k) = 1\}$. The average behaviour of $\delta_k(n)$ has been extensively studied by many authors[1-4, 9-12]. In this paper, we investigate the function

$$V(\lambda) \stackrel{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N} \sum_{n \le N,\, \delta_k(n) < n\lambda} 1 \qquad \ldots (1.1)$$

where $0 \le \lambda \le 1$.

More precisely, we prove the following result.

Theorem 1

$$V(\lambda) = \frac{\phi(k)}{k} \sum_{\substack{n=1 \\ \delta_k(n) < n\lambda}}^{\infty} \frac{\chi_k(n)}{n} = \frac{\phi(k)}{k} \sum_{\substack{n=1 \\ \delta_k(n) < n\lambda}}^{x} \frac{\chi_k(n)}{n} + O\left(\frac{\log^{r-1}(x)}{x}\right)$$

where $\phi(k)$ is Euler's totient and

$$\chi_k(n) = \begin{cases} 1, & \text{if every prime divisor of } n \text{ divides } k \\ 0, & \text{otherwise.} \end{cases}$$

2. Proof of Theorem 1

A theorem of Delange[6] (see also Elliott[7] for instance) says that if $g(n)$ is a multiplicative function such that (1) $|g(n)| \leq 1$, $n = 1, 2, \ldots$, and (2) the series $\Sigma_p (1 - g(p))/p$ converges then the limit

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} g(n) = M(g)$$

exists and

$$M(g) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{j=1}^{\infty} \frac{g(p^j)}{p^j}\right).$$

We apply the above theorem for $g(n) = \left(\frac{\delta_k(n)}{n}\right)^d$ where d is a natural number. Since

$$\sum_p \frac{1 - \left(\frac{\delta_k(p)}{p}\right)^d}{p} = \sum_{p \mid k} \frac{1}{p}\left(1 - \frac{1}{p^d}\right) < \infty,$$

condition (2) is satisfied. The other conditions of the theorem can be easily verified. Therefore,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \left(\frac{\delta_k(n)}{n}\right)^d$$

$$= \prod_{p \mid k} \left(1 - \frac{1}{p}\right)\left(1 + \sum_{j=1}^{\infty} \frac{1}{p^{(d+1)j}}\right) \prod_{p \nmid k} \left(1 - \frac{1}{p}\right)\left(1 + \sum_{j=1}^{\infty} \frac{1}{p^j}\right)$$

$$= \prod_{p \mid k} \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{p^{(d+1)}}\right)^{-1}. \quad \ldots (2.1)$$

We note that the arithmetical function $\chi_k(n)$, defined in the statement of Theorem 1, is completely multiplicative and

$$\sum_{n=1}^{\infty} \frac{\chi_k(n)}{n} = \prod_{p \mid k} \left(1 - \frac{1}{p}\right)^{-1} = \frac{k}{\phi(k)}.$$

Now,

$$\frac{\phi(k)}{k}\sum_{n=1}^{\infty}\frac{\chi_k(n)}{n}\left(\frac{\delta_k(n)}{n}\right)^d=\prod_{p\mid k}\left(1-\frac{1}{p}\right)\left(1+\sum_{j=1}^{\infty}\frac{1}{p^j}\left(\frac{\delta_k(p^j)}{p^j}\right)^d\right)$$

$$=\prod_{p\mid k}\left(1-\frac{1}{p}\right)\left(1-\frac{1}{p^{(d+1)}}\right)^{-1}. \qquad \text{... (2.2)}$$

Hence, by (2.1) and (2.2) we get

$$\frac{\phi(k)}{k}\sum_{n=1}^{\infty}\frac{\chi_k(n)}{n}\left(\frac{\delta_k(n)}{n}\right)^d=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\left(\frac{\delta_k(n)}{n}\right)^d.$$

This is true for all $d = 0, 1, 2, ...$

Consequently, for any polynomial $P(X)$,

$$\frac{\phi(k)}{k}\sum_{n=1}^{\infty}\frac{\chi_k(n)}{n}P\left(\frac{\delta_k(n)}{n}\right)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}P\left(\frac{\delta_k(n)}{n}\right).$$

Now, by using Weirstrass' Theorem on approximation of a continuous function by polynomials, we see that for any continuous function $F(X)$ on $[0, 1]$,

$$\frac{\phi(k)}{k}\sum_{m=1}^{\infty}\frac{\chi_k(n)}{n}F\left(\frac{\delta_k(n)}{n}\right)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}F\left(\frac{\delta_k(n)}{n}\right).$$

Next, taking the usual approximation of the function

$$S(X)=\begin{cases}1 & \text{if} \quad X\le\lambda\\ 0 & \text{if} \quad X>\lambda\end{cases}$$

by continuous functions, we obtain

$$\frac{\phi(k)}{k}\sum_{\substack{n=1\\ \delta_k(n)<n\lambda}}^{\infty}\frac{\chi_k(n)}{n}=\lim_{N\to\infty}\frac{1}{N}\sum_{\substack{n\le N\\ \delta_k(n)<n\lambda}}1=V(\lambda) \quad (\text{by} \quad (1.1)).$$

This proves the first part of the theorem.

Now, $$V(\lambda)=\frac{\phi(k)}{k}\sum_{\substack{n=1\\ \delta_k(n)<n\lambda}}^{x}\frac{\chi_k(n)}{n}+E(x), \qquad \text{... (2.3)}$$

where

$$0\le E(x)=\frac{\phi(k)}{k}\sum_{\substack{n>x\\ \delta_k(n)<n\lambda}}\frac{\chi_k(n)}{n}$$

$$\leq \sum_{n > x} \frac{\chi_k(n)}{n}$$

$$= \sum_{m=0}^{\infty} \sum_{2^m x < n \leq 2^{m+1} x} \frac{\chi_k(n)}{n}$$

$$\leq \sum_{m=0}^{\infty} \frac{1}{2^m x} \sum_{2^m x < n \leq 2^{m+1} x} \chi_k(n). \qquad \ldots (2.4)$$

The function

$$N(x) \overset{\text{def}}{=} \sum_{n \leq x} \chi_k(n) = \sum_{\substack{n = p_1^{i_1} \cdots p_r^{i_r} \\ n \leq x}} 1$$

counts the number of r-tuple of non-negative integers $(i_1, ..., i_r)$ satisfying

$$i_1 \log p_1 + \ldots + i_r \log p_r \leq \log x. \qquad \ldots (2.5)$$

In other words, $N(x)$ counts the number of integer lattice points in the tetrahedron defined by (2.5). We have the asymptotic formula (originally obtained for the two-dimensional case by Hardy and Littlewood[8], the general case being derivable by induction (Beukers[5]))

$$N(x) = c_1 (\log x)^r + c_2 (\log x)^{r-1} + o((\log x)^{r-1}) \qquad \ldots (2.6)$$

where

$$c_1 = \frac{1}{r! \log p_1 \ldots \log p_r}$$

and

$$c_2 = -\frac{1}{2((r-1)!)} \cdot \frac{\log p_1 + \ldots + \log p_r}{\log p_1 \ldots \log p_r}.$$

Therefore, from (2.4) and (2.6) we obtain that

$$0 \leq E(x) \leq \frac{1}{x} \sum_{m=0}^{\infty} \frac{1}{2^m} (N(2^{m+1} x) - N(2^m x))$$

$$= O\left(\frac{\log^{r-1}(x)}{x} \right).$$

This alongwith eqn. (2.3) completes the proof.

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