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A LOWER BOUND FOR CERTAIN RAMSEY TYPE PROBLEMS

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Abstract. One of the main results in this article is to present a new and simple proof of the following theorem; The Diophantine equation $a_1/y_1 + a_2/y_2 + \cdots + a_n/y_n = 0$, where $y_i \neq 0$ are n unknown variables and a_i's are non-zero integers, is regular if and only if there exists a non empty subset I of $\{1, 2, \cdots, n\}$ such that $\sum_{i \in I} a_i = 0$. In the rest of this article, we study a particular case when $n = 3$ in the above equation. Also, we define $f(r)$ to be the largest integer such that if the integers in $[1, f(r)]$ are r coloured, then no monochromatic solution to the Diophantine equation $1/x + 1/y = 1/z$, x, y, z can exist. Then we prove that $f(r) \geq 2^{r-3}.192 - 1$ and

$$f(r+2) \geq 4\left(f(r) + \sqrt{\frac{f(r)}{5}}\right).$$

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1. Introduction. A given Diophantine equation in n ($n \in \mathsf{N}$) variables $x_1, x_2, \cdots, x_n$ is said to be r-regular whenever the set of all positive integers N is r coloured in any manner, there exists a solution say $b_1, b_2, \cdots, b_n \in \mathsf{N}$ to that equation and all the b_i's having the same colour. Such a solution, we call to be monochromatic. If the Diophantine equation is r-regular for every natural number r, then it is said to be regular.

A classical theorem of Schur [7] states that the Diophantine equation $x + y = z$ is regular and in 1933, Rado [6] discovered the following, far more reaching generalization of the classical Schur's theorem.

Theorem of Rado. ([6]) *The linear homogeneous Diophantine equation $\sum_{i=1}^{n} a_i x_i = 0$, where $x_i \neq 0$'s are n unknown variables and a_i's are non-zero integers, is regular if and only if for some non-empty subset I of $\{1, 2, \cdots, n\}$ we have $\sum_{i \in I} a_i = 0$.*

Paul Erdős posed a question which is analogous to the classical Schur's theorem as follows:

"Is it true that the equation $1/x + 1/y = 1/z$ is regular?"

In 1991, this question was answered affirmatively, and in greater generality, by Lefmann in [5], as well as by Brown, and Rödl in [4]. Recently, Adhikari gave a new proof of Erdős's question (see [2]) by employing a method different from those of Lefmann, Brown, and Rödl; moreover, Adhikari's result answers the question for x, y, z distinct.

In Section 2, we shall consider the following Diophantine equation

$$\frac{a_1}{y_1} + \frac{a_2}{y_2} + \cdots + \frac{a_n}{y_n} = 0, \tag{1}$$

where the a_i's are non-zero integers and y_i's are n unknown variables in natural numbers. Here, we shall give a Rado-type criterion for this equation to be regular. Lefmann [5] proved this type of criterion in generality. Our proof is new and simpler for this particular type of equation.

Let $f(r)$ be the largest integer such that if the integers in $[1, f(r)]$ are r coloured, then no monochromatic solution x, y, z to the equation

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{z} \tag{2}$$

can exist. Prof. Schinzel asked the question (during a conference held at the university of Limoges) of finding a lower bound for $f(r)$ in the spirit of the result of Schur and its improvement by Abbott-Hanson [1]. In [3], it had been proved that $f(2) = 59$ and $f(r+2) \geq 4f(r) + 3$.

In Section 3, we deal with this question of finding a lower bound which improves the above mentioned results. Indeed, we could obtain the following results in this regard.

THEOREM 3.1. $f(3) \geq 191$.

THEOREM 3.2. *For each $r \in \mathbb{N}$, we have*

$$f(r+2) \geq 4\left(f(r) + \sqrt{\frac{f(r)}{5}}\right).$$

THEOREM 3.3. *For every intger $r \geq 3$, we have*

$$f(r) \geq 2^{r-3} \cdot 192 - 1.$$

In the last section we shall discuss some known upper bounds of $f(r)$ and some open questions.

2. A criterion of Rado type.

THEOREM 2.1. *The Diophantine equation*

$$\frac{a_1}{y_1} + \frac{a_2}{y_2} + \cdots + \frac{a_n}{y_n} = 0,$$

where the a_i's are non-zero integers and the y_i's are n unknown variables in natural numbers, is regular if and only if there exists some non-empty subset of the a_i's which sums to 0.

Proof. Suppose there exists a non-empty subset of the a_i's which sums to 0. Then, by the theorem of Brown and Rödl (see Theorem 2.1, [4]), we get the equation (1) is regular.

Conversely, suppose that (1) is regular, and suppose, for proof by contradiction, that all the non-empty sums of the a_i's are non-zero. Let p be some prime which does not divide any of these non-empty sums, and consider the $(p-1)$-colouring of the natural numbers which assigns to $n = p^a q$, $\gcd(p, q) = 1$, the colour $q \pmod p$. Denote the colour assigned to n by $\Psi(n)$.

By hypothesis, (1) has a monochromatic solution $(c_1, ..., c_n)$, where $c_i = p^{\ell_i} m_i$, gcd $(m_i, p) = 1$, and where all the m_i's are congruent to a common number β $\pmod p$, $1 \leq \beta \leq p-1$. Set $\ell = \max_i\{\ell_i\}$, and suppose, without loss of generality, that $\ell_1 = \ell$. Then,

$$0 \;=\; p^\ell \beta \sum_{i=1}^{n} \frac{a_i}{c_i} \;=\; \sum_{i=1}^{n} a_i p^{\ell-\ell_i} \beta m_i^{-1} \equiv \sum_{i=1}^{n} a_i p^{\ell-\ell_i} \pmod p.$$

The sum on the right hand side is congruent, modulo p, to a sum of the a_i's, and this sum is non-empty because it includes a_1. This contradicts our assumption that p does not divide any of the non-empty sums of the a_i's, and so our theorem follows. □

REMARK 2.2. Theorem 2.1 tells that the equation $1/z = 1/x + 1/y$ is regular whereas Erdős-Strauss' equation $4/x_4 = 1/x_1 + 1/x_2 + 1/x_3$ and Sierpiński's equation $5/x_4 = 1/x_1 + 1/x_2 + 1/x_3$ are not regular.

3. A lower bound. Before we go into the proof of Theorem 3.1, we fix up a notation as follows. If m and n are two natural numbers, then m#n means the set of natural numbers $m, m+1, \cdots, n$ whenever $m \leq n$ or m#n means $m, m-1, \cdots, n$ whenever $m \geq n$.

THEOREM 3.1. $f(3) \geq 191$.

Proof. To prove this, we have to produce a 3-colouring

$$\{1, 2, \cdots, 190, 191\} = C_1 \cup C_2 \cup C_3$$

such that none of the C_i's contains a solution of the equation (2). Let us define the colouring as follows;

$$C_1 = \{1, 3, 5, 7, 8, 11, 24\#47\},$$

$$C_2 = \{4, 6, 10, 13, 14, 18, 48\#95\}$$

and

$$C_3 = \{2, 9, 12, 15, 16, 17, 19, 20\#23, 96\#191\}.$$

The idea is the following. First, colour the integers from 191 to 24 (in the descending order) using 3 colours as the integer 191#96 gets one colour, the integers 95#48 gets another and the integers 47#24 gets the third.

Note that the integers in the each subset having the same colour lie between n and $(n+1)/2$ for some integer n. So, none of the subsets contains a monochromatic solution of the equation (2). Now, we left with the integers 1 to 23. We just distribute them in one of the subsets so that none of the subsets contains a monochromatic solution. Hence the Theorem. □

REMARK 3.1.1. Note that with the above idea, we are unable to extend further. For example, if we colour the following three blocks 193#97, 96#49 and 48#25 into three different colours, then we are unable to put 24 into any of the classes so that none of the classes contains a monochromatic solution to the equation (2). For instance, with the above colouring, we have a monochromatic solution $1/18 = 1/24 + 1/72$.

THEOREM 3.2. *For each* $r \in \mathbb{N}$, *we have*

$$f(r+2) \geq 4\left(f(r) + \sqrt{\frac{f(r)}{5}}\right).$$

Proof. To prove this, it suffices to produce an $(r+2)$-colouring of the integers in $[1, 4(f(r) + \sqrt{f(r)/5})$ which avoids a solution to (2). First, colour the integers $[1, f(r)]$ with the first r colours, so as to avoid a solution to (2). Colour the integers in $I_1 \cup I_2$ the $(r+1)^{\mathrm{st}}$ colour, where

$$\begin{aligned} I_1 &= \left[f(r)+1,\ f(r)+\sqrt{f(r)/5}\right), \\ I_2 &= \left[2(f(r)+\sqrt{f(r)/5}),\ 4(f(r)+\sqrt{f(r)/5})\right), \end{aligned}$$

and finally, colour the integers in

$$I_3 = \left[f(r)+\sqrt{f(r)/5},\ 2f(r)+2\sqrt{f(r)/5}\right)$$

with the $(r+2)^{\mathrm{nd}}$ colour. With this colouring, there can be no solution involving the $(r+2)^{\mathrm{nd}}$ colour, since if $x, y, z \in I_3$ is a solution to (2), then we have

$$\frac{1}{x} + \frac{1}{y} > \frac{2}{2(f(r)+\sqrt{f(r)/5})} = \frac{1}{f(r)+\sqrt{f(r)/5}} \geq \frac{1}{z}.$$

By a similar argument, one can show that a solution involving the $(r+1)^{\mathrm{st}}$ colour must have $y, z \in I_1$ and $x \in I_2$ (or, $x, z \in I_1$ and $y \in I_2$ – assume, without loss of generality, that $y, z \in I_1$).

Note that since $x, y \geq 1$ and $1/x + 1/y = (x+y)/xy$, which implies whenever x or y is equal to $xy/(x+y)$, then we get y or x is equal to zero which is a contradiction. Hence $x \neq z$ and $y \neq z$. Let $g = \gcd(y, z)$ and set $d_1 = z/g$ and $d_2 = y/g$. Notice that $\gcd(d_1, d_2) = 1$. We will now show that $d_1, d_2 > \sqrt{5f(r)}$: since $g||z-y| < \sqrt{f(r)/5}$, and since $z \neq y$, we must have that $g < \sqrt{f(r)/5}$, which implies that

$$d_1, d_2 \geq \frac{f(r)+1}{g} > \sqrt{5f(r)}.$$

Thus,

$$\frac{1}{x} = \frac{1}{z} - \frac{1}{y} = \frac{1}{g}\left(\frac{1}{d_1} - \frac{1}{d_2}\right) = \frac{d_2 - d_1}{g d_1 d_2},$$

and $(d_1 - d_2, d_1 d_2) = 1$. Therefore, $x \geq d_1 d_2 > 5f(r)$, which is a contradiction. This proves the Theorem. □

THEOREM 3.3. *For every integer $r \geq 3$, we have $f(r) \geq 2^{r-3}.192 - 1$.*

Proof. First, we shall prove that $f(r+1) \geq 2f(r)+1$. Indeed, it suffices to produce an $(r+1)$-colouring of the integers in $[1, 2f(r)+1]$ which avoids a solution to (2). First, colour the integers in $[1, f(r)]$ with the first r colours so as to avoid a solution to (2), and then colour all the integers in $[f(r)+1, 2f(r)+1]$ the $(r+1)^{\mathrm{st}}$ colour. Any solution to (2) must have $x, y, z \in [f(r)+1, 2f(r)+1]$, but this is impossible, since

$$\frac{1}{x} + \frac{1}{y} \geq \frac{2}{2f(r)+1} > \frac{1}{f(r)+1} \geq \frac{1}{z}.$$

Now, to prove the theorem, we shall do the following: Since $f(r+1) \geq 2f(r)+1$, we get, by iteration,

$$f(r) \geq 2^{r-3} f(3) + 2^{r-4} + 2^{r-5} + \cdots + 2^2 + 2 + 1 = 2^{r-3}(f(3)+1) - 1.$$

Since $f(3) \geq 191$ by Theorem 3.1, we get $f(r) \geq 2^{r-3} \cdot 192 - 1$. □

REMARK 3.3.1. (i) The set of all odd positive integers contains no solution of the equation (2) because the numerator of the sum $1/x + 1/y$ has a factor of 2 (as x and y are odd) which cannot get cancelled.

(ii) Also, note that $f(r)+1$ cannot be an odd prime number. Assume that it is an odd prime number p. We know, by definition of $f(r)$, there exists an r-colouring of the integers in $[1, f(r)]$ which has no monochromatic solution to the equation (2). More precisely, there exists $C_1, C_2, \cdots, C_r$ such that

$$[1, f(r)] = C_1 \cup C_2 \cup \cdots \cup C_r$$

and none of the C_is contains a solution to the equation (2). If we include the prime number p into any of the C_i, then there is a solution to the equation (2) involving

p. Also note that, in the equation (2), $z \neq p$, since p is the largest integer in that set. Therefore, p is either x or y. Thus assuming $y = p$, we get, $1/x + 1/p = 1/z$ where $x, z < p$. This implies, $xz = p(x - z)$. It is impossible, since $x, z < p$.

(iii) It may happen that $f(r)$ is odd for every integer $r \geq 2$.

THEOREM 3.4. *If $f(r)$ is odd, then we have $f(r+1) \geq 2f(r) + 3$.*

Proof. It suffices to colour the integers in $[1, 2f(r) + 3]$ by $(r + 1)$ colours such that this colouring has no monochromatic solution to (2). Here is such a colouring: colour the even integers in $[2, 2f(r)]$ with the first r colours, so as to avoid a monochromatic solution (upon dividing these integers by 2, one can see this is the same way as r-colouring the integers in $[1, f(r)]$ to avoid monochromatic solutions), and then colour $2f(r) + 2$ and the odd integers in $[1, 2f(r) + 3]$ by the $(r + 1)^{\text{st}}$ colour. Any monochromatic solution must have x, y, and z all being of the $(r+1)^{\text{st}}$ colour, so let us assume x, y, and z are coloured by the $(r + 1)^{\text{st}}$ colour.

Now, x, y, and z cannot all be odd, by above Remark 3.3.1 (i); moreover, z must be odd, since otherwise

$$\frac{1}{z} = \frac{1}{2f(r) + 2} < \frac{2}{2f(r) + 3} \leq \frac{1}{x} + \frac{1}{y}.$$

If x and y were both even, then $1/x + 1/y = 2/(2f(r) + 2) = 1/(f(r) + 1)$, which is not coloured with the $(r + 1)^{\text{st}}$ colour. So, we conclude that one of x or y is even, while the other one is odd, which is impossible, since if $1/x + 1/y = 1/z$, then $2|z$. We conclude that no monochromatic solutions exist for our colouring, and hence the theorem. $\square$

4. Concluding remarks. Let $S(r)$ denote the classical Schur number, which is defined to be the smallest positive integer such that if $\{1, 2, \cdots, S(r)\}$ is r coloured, then there exists a monochormatic solution to $x + y = z$. In [7], Schur proved that $S(r) \leq [er!]$ and some slight improvements have since appeared. Our interest is to connect this number with $f(r)$. In 1991, Brown and Rödl [4] while proving the existence of the monochromatic solution of the equation (2), they argued as follows; If we put $T(r)$ to be the least common multiple (lcm) of $\{1, 2, \cdots, S(r)\}$ where $S(r)$ is the Schur number, then they proved that given any r colouring of the set $\{1, 2, \cdots, T(r)\}$, we have a monochromatic solution of the equation (2). This implies, in our notation $f(r) \leq T(r) - 1$. Since $S(3) = 16$ (for instance, see [8]), we have $T(3) = \text{lcm}\{1, 2, \cdots, 16\} = 720720$. Therefore, $f(3) \leq 720719$, which is seemingly too big.

Let $N = n(2, 3, r)$ and $M_r = P(N)p_N p_{N-1}/2$ where p_m denotes the mth prime, $n(2, 3, r)$ is the Ramsey number and $P(N)$ is the lcm of all positive differences of p_m for all $m = 1, 2, \cdots, N$. Then, in Adhikari's [2] method one can arrive at the first distinct solution to the equation (2) in the interval $[1, M_r]$. But M_3 is too large when compared to T_3. Thus, at present we know that $191 \leq f(3) \leq 720719$.

From the Theorem 3.4, one notes that $f(3) \geq 3f(2) + 2 = 3 \cdot 59 + 2 = 179$. We conclude this article by raising the following questions;

(i) "Is it true for every natural number $r \geq 1$ that $f(r + 1) \geq 3f(r) + 2$?"

(ii) "Is it true for every natural number $r \geq 1$ that $f(r)$ is odd?"

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