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Expo. Math. 23 (2005) 171–177

EXPOSITIONES
MATHEMATICAE

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On Steinhaus sets

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Received 19 February 2004

Abstract

We give a common proof of several results on Steinhaus sets in $\mathbb{R}^d$ for $d \geq 2$ including the fact that a Steinhaus set in $\mathbb{R}^2$ must be disconnected.

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MSC 2000: primary 11P21; secondary 52C15

Keywords: Steinhaus sets; Integer lattice points

1. Introduction

A subset S of $\mathbb{R}^d$ is said to be a Steinhaus set if $|(\rho \cdot S + x) \cap \mathbb{Z}^d| = 1$ for any $\rho \in SO(d)$ and any $x \in \mathbb{R}^d$, where $\rho \cdot S = \{\rho x : x \in S\}$. So, S is a Steinhaus set if every rigid motion of S contains exactly one integer lattice point in $\mathbb{R}^d$.

In 1957, Polish mathematician H. Steinhaus asked whether there exists a Steinhaus set in $\mathbb{R}^2$ (cf. [10]). The existence of a Steinhaus set in dimension 2 has been shown recently by Jackson and Mauldin (cf. [5–7]). Whether there exists a Steinhaus set in higher dimension is still an open question. Steinhaus sets have been the subject of several recent papers. In this note, we give a common proof of some results (which are known in $\mathbb{R}^2$) and some new results on Steinhaus sets in any dimension.

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We fix some notation first. The point $(x_1, x_2, \dots, x_d)$ in $\mathbb{R}^d$ will be simply denoted by x . Let $a_1 < a_2 < a_3 < \dots$ be the increasing enumeration of all non-zero integers which can be written as a sum of d squares. For any $x \in \mathbb{R}^d$ and $\varepsilon > 0$, $B(x, \varepsilon)$ will denote the open ball in $\mathbb{R}^d$ with center x and radius ε , $C(x, \varepsilon)$ the closed ball in $\mathbb{R}^d$ with center x and radius ε , $\mathbb{S}(x, \varepsilon)$ the sphere in $\mathbb{R}^d$ with center x and radius ε and $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$; $A_n(x, \varepsilon)$ will denote the annulus

$$A_n(x, \varepsilon) = \{z \in \mathbb{R}^d : \sqrt{a_n} - \varepsilon < |z - x| < \sqrt{a_n} + \varepsilon\}.$$

Theorem 1. *Let $S \subset \mathbb{R}^d$ be a Steinhaus set, $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Then $S \cap A_n(x, \varepsilon) \neq \emptyset$ for some $n \geq 1$.*

Remark. The method of the proof of this theorem is essentially derived from the result of Ciucu [3] (see Corollary 4 in this article). Indeed, Theorem 1 is equivalent to his result which says that the interior of a Steinhaus set is empty.

As a corollary, we show that a Steinhaus set in $\mathbb{R}^d$ does not contain a homeomorph of $\mathbb{S}^{d-1}$ (Corollary 5). We then prove that a Steinhaus set is disconnected. It is still not known whether a Steinhaus set is always totally disconnected.

2. Proof of our results

We shall start this section with following lemma.

Lemma 2. *Let $b_1, b_2, \dots$ be any sequence in $\mathbb{R}$ such that $b_n \rightarrow \infty$ as $n \rightarrow \infty$. Then for any $\varepsilon > 0$, there exists a positive integer N such that for every $n \geq N$ there exists an integer ℓ of the form $\ell = r^2 + t^2$ satisfying $b_n < \ell < b_n + 2\varepsilon\sqrt{b_n}$.*

Proof. Let $x = b_n$ and $t = [\sqrt{x}]$ where $[x]$ denotes the integral part of x . So, $t = \sqrt{x} - \theta$ for $0 \leq \theta < 1$. Let x_1 and x_2 satisfy

$$x = x_1^2 + t^2 \tag{1}$$

and

$$x + 2\varepsilon\sqrt{x} = x_2^2 + t^2. \tag{2}$$

Subtracting Eq. (1) from Eq. (2), we get

$$x_2 - x_1 = \frac{2\varepsilon\sqrt{x}}{x_1 + x_2}. \tag{3}$$

Now,

$$x_1 = \sqrt{x - t^2} = \sqrt{x - (\sqrt{x} - \theta)^2} = \sqrt{2\theta\sqrt{x} - \theta^2} < \sqrt{2} x^{1/4} \tag{4}$$

and

$$x_2 = \sqrt{x + 2\varepsilon\sqrt{x} - t^2} = \sqrt{2\varepsilon\sqrt{x} + 2\sqrt{x}\theta - \theta^2} < \sqrt{2(\varepsilon + 1)x^{1/4}}. \quad (5)$$

Therefore from Eqs. (4) and (5), we get

$$x_1 + x_2 < \left(\sqrt{2}(1 + \sqrt{\varepsilon + 1})\right)x^{1/4}.$$

Substituting this in Eq. (3), we get

$$x_2 - x_1 > \frac{2\varepsilon\sqrt{x}}{\left(\sqrt{2}(1 + \sqrt{\varepsilon + 1})\right)x^{1/4}} = \frac{\sqrt{2}\varepsilon}{1 + \sqrt{\varepsilon + 1}}x^{1/4}.$$

Since $b_n \rightarrow \infty$ as $n \rightarrow \infty$, there exists a positive integer N such that for every $n \geq N$, $x_2 - x_1 > 2$. Thus for every $n \geq N$, there exists a positive integer r such that $x_1 < r < x_2$. Therefore, $x_1^2 + t^2 < r^2 + t^2 < x_2^2 + t^2$. So $b_n < \ell = r^2 + t^2 < b_n + 2\varepsilon\sqrt{b_n}$. $\square$

Corollary 3. $\sqrt{a_{n+1}} - \sqrt{a_n} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. Clearly, $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Note that

$$\sqrt{a_{n+1}} - \sqrt{a_n} = \frac{a_{n+1} - a_n}{\sqrt{a_{n+1}} + \sqrt{a_n}} < \frac{a_{n+1} - a_n}{2\sqrt{a_n}}.$$

By Lemma 2, for every $\varepsilon > 0$, there is a positive integer N such that for every $n \geq N$, there exist integers r and t satisfying $a_n < r^2 + t^2 < a_n + 2\varepsilon\sqrt{a_n}$. Therefore, by the above inequality, $\sqrt{a_{n+1}} - \sqrt{a_n} < \varepsilon$ for every $n \geq N$. $\square$

Remark. It is known by Lagrange's theorem that every positive integer can be written as sum of four squares. Therefore, Corollary 3 trivially holds for $d \geq 4$, since $\sqrt{n+1} - \sqrt{n} < 1/(2\sqrt{n})$, we have $\sqrt{a_{n+1}} - \sqrt{a_n} = \sqrt{n+1} - \sqrt{n} \rightarrow 0$ as $n \rightarrow \infty$.

Proof of Theorem 1. Replacing S by $S - x$, we assume that $x = 0$, the origin. For easy notation, we shall write A_n instead of $A_n(0, \varepsilon)$. If possible, suppose $S \cap A_n = \emptyset$ for every n .

By Corollary 3, there exists a positive integer N such that $[\cup_{n \geq N} A_n]^c$ is a closed ball $C(0, R)$ with center the origin and radius R . So, by our hypothesis, $S \subset C(0, R)$. Thus S is bounded.

For any $\bar{m} = (m_1, \dots, m_d) \in \mathbb{Z}^d$, we denote,

$$I_{\bar{m}} = \prod_{i=1}^d [m_i, m_i + 1).$$

Let

$$F = \left\{ \bar{m} \in \mathbb{Z}^d : \bar{m} \neq 0, \quad \exists \rho \in SO(d) \quad \text{and } x \in I_0 \text{ such that } x + \bar{m} \in \rho \cdot S \right\}.$$

Since S is bounded, F is finite. Set

$$I(\varepsilon) = \bigcap_{\bar{m} \in F} [(A_n \cap I_{\bar{m}}) - \bar{m}],$$

where $m_1^2 + \cdots + m_d^2 = a_n$. Note that $I(\varepsilon) \subset I_0$.

Claim 1. $I(\varepsilon) \subset \rho \cdot S$ for every $\rho \in SO(d)$.

Let $\rho \in SO(d)$ and $x \in I(\varepsilon)$. Since $\rho \cdot S$ is a Steinhaus set, there is a $\bar{m}$ such that $x + \bar{m} \in \rho \cdot S$. If $\bar{m} \neq 0$, then by definition of F , $\bar{m} \in F$. But this implies that $x + \bar{m} \in A_n$, where $m_1^2 + \cdots + m_d^2 = a_n$. Thus, $A_n \cap [\rho \cdot S] \neq \emptyset$. This contradicts our assumption that $S \cap A_n = \emptyset$. Thus, $\bar{m} = 0$ and hence $x \in \rho \cdot S$.

Let

$$B = \bigcup_{\rho \in SO(d)} \rho \cdot I(\varepsilon).$$

By Claim 1, it follows that $B \subset S$. It is easy to see that there exists a $\delta > \varepsilon$ such that $B(0, \delta) \subset B \subset S$. Let δ be the largest number such that $B(0, \delta) \subset S$.

Claim 2. For every $n \geq 1$, $S \cap A_n(0, \delta) = \emptyset$.

If possible, suppose for some $n \geq 1$, $S \cap A_n(0, \delta) \neq \emptyset$. Let $y \in S \cap A_n(0, \delta)$. Clearly, there exists a $z \in B(0, \delta)$ such that $|y - z| = \sqrt{a_n}$. Since $B(0, \delta) \subset S$, this implies that there exists $y, z \in S$ such that $|y - z| = \sqrt{a_n}$. But this contradicts the fact that S is a Steinhaus set. Thus the claim follows.

Set

$$I(\delta) = \bigcap_{\bar{m} \in F} [(A_n(0, \delta) \cap I_{\bar{m}}) - \bar{m}],$$

where $m_1^2 + \cdots + m_d^2 = a_n$ (and F is as defined before).

By the argument contained in the proof of Claim 1, we see that $I(\delta) \subset S$. Also, by the argument following the proof of Claim 1,

$$\cup_{\rho} \rho \cdot I(\delta) \subset S$$

and hence we see that there exists a $\delta' > \delta$ such that $B(0, \delta') \subset I(\delta)$. This contradicts the maximality of δ and our proof is complete. $\square$

Corollary 4. Every Steinhaus set S has empty interior.

Proof. If possible, suppose the interior of S (denoted by S°) is non-empty. Assume without loss of any generality that $0 \in S^\circ$. Get a $\delta > 0$ such that $B(0, \delta) \subset S$. By Claim 2 of Theorem 1, we see that $S \cap A_n(0, \delta) = \emptyset$ for every $n \geq 1$. This contradicts Theorem 1. $\square$

Corollary 4 implies Theorem 1. Assume that the interior of S is empty. If $S \cap A_n(0, \varepsilon) = \emptyset$ for all $n \geq 1$, then $B(0, \varepsilon) \subset S$. For, if $x \in B(0, \varepsilon) \setminus S$, then there is $y \in S$ with $|x - y| = a_n$

for some n . But then $||y| - a_n| < \varepsilon$ and so $y \in A_n(0, \varepsilon)$ which is a contradiction. Thus, $B(0, \varepsilon) \subset S$ which contradicts the assumption that interior of S empty. $\square$

The following corollary is the generalization of the result in [1].

Corollary 5. *No Steinhaus set S in $\mathbb{R}^d$ contains a homeomorph of $\mathbb{S}^{d-1}$.*

Proof. If possible, suppose there exists a Steinhaus set S in $\mathbb{R}^d$ containing a homeomorph C of $\mathbb{S}^{d-1}$. By the Jordan curve theorem (cf. [11, Corollary 36.4]), $\mathbb{R}^d \setminus C$ has a bounded component E . Without any loss of generality, we assume that $0 \in E$. Get a $\delta > 0$ such that $B(0, \delta) \subset E$.

By Theorem 1, our result will follow, if we show that $S \cap A_n(0, \delta) = \emptyset$ for every n . Suppose for some n , $S \cap A_n(0, \delta) \neq \emptyset$. Take any $x \in S \cap A_n(0, \delta)$. By looking at the intersection of S with the line passing through 0 and x , we see that there exist y and z in C such that $|x - y| < \sqrt{a_n}$ and $|x - z| > \sqrt{a_n}$. Since C is connected, there must be a point $p \in C$ such that $|x - p| = \sqrt{a_n}$. But this is impossible because S is Steinhaus. $\square$

Using Corollary 4 we now deduce some results.

Corollary 6. *No closed set S is Steinhaus.*

Proof. Suppose S is a closed Steinhaus set in $\mathbb{R}^d$. Since S is a Steinhaus set,

$$\mathbb{R}^d = \bigcup_{\bar{m} \in \mathbb{Z}^d} (S + \bar{m}).$$

As S is closed, each $S + \bar{m}$ is closed. So, by the Baire Category theorem, $S + \bar{m}$ has non-empty interior for some $\bar{m}$, which implies S has non-empty interior. This is impossible by Corollary 4. Hence S cannot be closed. $\square$

It has been already proved that no Steinhaus set S in $\mathbb{R}^2$ has the property of Baire (cf. [5]). Their proof can be easily modified to see that this result is true in any dimension. In [2,4], it is shown that no bounded Steinhaus set in $\mathbb{R}^2$ is Lebesgue measurable, whereas in [8,9], it is shown that there does not exist a Lebesgue measurable Steinhaus set in $\mathbb{R}^d$ for any $d \geq 3$. The existence of an unbounded Lebesgue measurable Steinhaus set in $\mathbb{R}^2$ is still open.

Lemma 7. *If S is a connected Steinhaus set in $\mathbb{R}^d$, then there does not exist $z_1, z_2 \in S$ such that $|z_1 - z_2| > 1$.*

Proof. Suppose S is a connected Steinhaus set in $\mathbb{R}^d$. We shall arrive at a contradiction. Without loss of generality we assume that the origin $0 \in S$. Since S is a Steinhaus set, for every $x \in S$, $S \cap \mathbb{S}(x, 1) = \emptyset$. As S is connected, it follows that $S \subset B(x, 1)$ for every $x \in S$. Therefore, $S \subset B(0, \frac{1}{2})$. For otherwise, there exists $z_1, z_2 \in S$ such that $|z_1 - z_2| > 1$. Hence, $S \subset B(z_1, 1) \cap B(z_2, 1)$ implies $z_1, z_2 \notin S$ which is a contradiction. Hence, there does not exist $z_1, z_2 \in S$ such that $|z_1 - z_2| > 1$. $\square$

Theorem 8. No Steinhaus set S in $\mathbb{R}^d$ is connected.

We shall present two proofs of this theorem. One uses Corollary 4 and the other is self-contained.

Let S be a connected Steinhaus set in $\mathbb{R}^d$. Without loss of generality we may assume that S contains the origin $0 \in S$. By Lemma 7, it follows in particular that $S \subset B(0, 1)$.

Proof No. 1. First we claim the following. If $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^d$, then, $e_1 \in \bar{S}$ (the closure of S).

For, if not, let $\varepsilon > 0$ be such that $S \cap B(e_1, \varepsilon) = \emptyset$. Set

$$A = B(0, 1) \cap B(e_1, \varepsilon)$$

and

$$J = \left\{ \bar{m} \in \{0, 1\}^d : m_1 = 1, \exists i > 1 \text{ such that } m_i = 1 \right\}.$$

Now consider the non-empty open set

$$E = \bigcap_{\bar{m} \in J} \{ \bar{x} \in A : x_i > 0 \quad \forall i \text{ and } |\bar{x} - \bar{m}| > 1 \}.$$

Let $z \in E$ be an arbitrary element. Since S is a Steinhaus set, $z + \bar{m} \in S$ for some $\bar{m} \in \mathbb{Z}^d$. As $S \subset B(0, 1)$ and $z \in E$, we must have $\bar{m} = -e_1$. Thus, $E - e_1 \subset S$. In particular, S has non-empty interior. This is a contradiction by Corollary 4. Thus our claim is proved.

Since e_1 is a limit point of S , there is a sequence $\{x_n\}$ in S such that $x_n \rightarrow e_1$. Since $S \subset B(x_n, 1)$ for each n , we have $S \subset C(e_1, 1)$. By the same argument, we see that $S \subset C(-e_1, 1)$ also.

Thus $0 \in S \subset B(0, 1) \cap C(e_1, 1) \cap C(-e_1, 1)$. Hence $S = \{0\}$. This is clearly not a Steinhaus set and our proof is complete. $\square$

Proof No. 2. First we show that for every $z = (x_1, x_2, \dots, x_d) \in S$, one of the following segments:

$$V_i = \prod_{j=1}^{i-1} \{x_j\} \times \left[x_i, x_i + \frac{1}{4} \right] \times \prod_{j=i+1}^d \{x_j\},$$

$i = 1, 2, \dots, d$, is contained in S .

Suppose not. That is, there exists $z \in S$ such that the above claim is not true. Without loss of generality we may assume that $z = 0$ the origin. Then there exists $z_i \in V_i \setminus S$ for each $i = 1, 2, \dots, d$. Since S is a Steinhaus set, S must meet $\mathbb{Z}^d + z_i$ for each i . Since $S \subset B(0, 1)$, in fact, S must meet at the point

$$w_i = (\underbrace{0, 0, \dots, 0}_{i-1 \text{ times}}, -1, \underbrace{0, 0, \dots, 0}_{d-i \text{ times}}) + z_i, \quad \forall i,$$

because all the other points of $\mathbb{Z}^d + z_i$ have distance > 1 from the origin. Then, we see that $|w_i - w_j| \geq \frac{3}{4}\sqrt{2} > 1$ for every $i \neq j$ which is not possible by Lemma 7. Therefore our claim is true.

Now take any point $x \in S$. By the above observation, there is another point $y \in S$ with $|x - y| > 1$ such that there is a polygonal path in S from x to y . This again contradicts Lemma 7. $\square$

Remark. Referee pointed out that Steinhaus set cannot exist in $\mathbb{R}^d$ for $d \geq 4$. If one replace $\mathbb{Z}^d$ by other lattice L in $\mathbb{R}^d$ in the definition of Steinhaus set, one may possibly obtain similar results that of the above with respect to L .

References

- [1] S.D. Adhikari, R. Balasubramanian, R. Thangadurai, Further remarks on Steinhaus sets, *Publ. Math. Debrecen* 57 (3–4) (2000) 277–281 (Corrigenda to: Further remarks on Steinhaus sets, *Publ. Math. Debrecen* 60(1–2) (2002) 209–210).
- [2] J. Beck, On a lattice point problem of H. Steinhaus, *Studia Sci. Math. Hungary* 24 (1989) 263–268.
- [3] M. Ciucu, A remark on sets having the Steinhaus property, *Combinatorica* 16 (3) (1996) 321–324.
- [4] H.T. Croft, Three lattice point problems of Steinhaus, *Quart. J. Math. Oxford Ser. (2)* 33 (1981) 71–83.
- [5] S. Jackson, R.D. Mauldin, On a lattice problem of H. Steinhaus, *J. Amer. Math. Soc.* 15 (2002) 817–856.
- [6] S. Jackson, R.D. Mauldin, Sets meeting isometric copies of the lattice $\mathbb{Z}^2$ in exactly one point, *Proc. National Acad. Sci.* 99 (25) (2002) 15883–15887.
- [7] S. Jackson, R.D. Mauldin, Survey of the Steinhaus tiling problem, *Bull. Symbolic Logic* 9 (3) (2003) 335–361.
- [8] M. Kolountzakis, M. Papadimitrakakis, The Steinhaus tiling problems and the range of certain quadratic forms, *Illinois J. Math.* 46 (3) (2002) 947–951.
- [9] M. Kolountzakis, T. Wolff, On the Steinhaus tiling problem, *Mathematika* 46 (2) (1999) 253–280.
- [10] W. Moser, *Research Problems in Discrete Geometry*, sixth ed., McGill University, Montréal, 1981.
- [11] J.R. Munkres, *Elements of Algebraic Topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.