

ON THE DIOPHANTINE EQUATION $ax^3 + by + c = xyz$

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Abstract: Consider the diophantine equation $ax^3 + by + c = xyz$, where a, b and c are positive integers such that $\gcd(a, c) = 1$ and c is square-free. Let (x, y, z) be a positive integral solution of the equation. In this paper, we shall give an upper bound for x, y and z in terms of the given inputs a, b and c . Also, we apply our results to investigate the divisors of the elements of the sequence $\{an^3 + c\}$ in residue classes.

Keywords: ?.

1. Introduction

Consider the diophantine equation

$$ax^3 + by + c - xyz = 0, \quad (1)$$

where x, y and z are unknown positive integers and, a, b and c are fixed positive integers such that $\gcd(a, c) = 1$ and c is square-free. This equation has been studied by many authors including Mohanty [4], Utz [10], Mohanty-Ramasamy [5] and [6], Luca-Togbé [3], Togbé [9], Subburam [7], Subburam-Thangadurai [8], etc.. In 1996, Mohanty-Ramasamy in [6] proved that there are only finitely many integral solutions to (1).

Let $N(a, b, c)$ denotes the number of positive integral solutions (x, y, z) of equation (1). By the result of Mohanty-Ramasamy in [6], it is known that $N(a, b, c)$ exists and it is finite. Recently, Subburam-Thangadurai [8] produced upper bounds for x, y and z , where (x, y, z) is a positive integral solution of equation 1 when $a = 1 = c$ and investigated the divisors of the element of the sequence $\{n^3 + 1\}$ in residue classes modulo n . In this paper, we give upper bounds for x, y and z of equation (1) in terms of a, b and c . Also, by an application of this result, we study the divisors of the elements of the sequence $\{an^3 + c\}$ in residue classes modulo n .

Theorem 1. *Any positive integral solution (x, y, z) of (1) satisfies*

$$x \leq abc^6 [a^2b^3c^8(a^2b^2c^{11} + a^2bc^{11} + 1)^3 + 1] + c^2,$$

$$y \leq ac^6 [a^2b^3c^8(a^2b^2c^9 + a^2bc^{11} + 1)^3 + 1]$$

and

$$z \leq ac^3 \{abc^5 [a^2b^3c^8(a^2b^2c^9 + a^2bc^{11} + 1)^3 + 1] + c\}^2 + bc + c^2.$$

From Theorem 1, we write the following corollary.

Corollary 1. *Let $M = \max\{a, b, c\}$. Then any positive integral solution (x, y, z) of (1) satisfies*

$$\max\{x, y, z\} \leq 3^9 M^{128}.$$

Theorem 2. *We have*

$$\sum_{n=1}^{\infty} \sum_{\substack{d \mid an^3 + c \\ d \equiv -b \pmod{n}}} 1 = N(a, b, c).$$

In 1984, H. W. Lenstra [2] proved:

For every real number $\alpha > 1/4$, there exists a constant $\kappa(\alpha)$ with the following property. If r, s and N are integers such that $0 \leq r < s < N$, $s > N^\alpha$ and $\gcd(r, s) = 1$, then there are at most $\kappa(\alpha)$ positive divisors of N which are congruent to r modulo s .

Also, in the same paper, he showed that if $\alpha > 1/3$, then $\kappa(\alpha) = 11$. In 2007, Coppersmith *et al* [1] showed that if $\alpha > 0.331$, then $\kappa(\alpha) = 32$. From this result, we can prove that if $n > 2^{48} \max\{a, b\}^{48}$ and b are any positive integers, then

$$\sum_{\substack{d \mid an^3 + c \\ d \equiv -b \pmod{n}}} 1 \leq 32.$$

As an immediate consequence of Theorems 1 and 2, we get the following corollary.

Corollary 2. *Let $M = \max\{a, b, c\}$. Then we have*

$$\sum_{\substack{d \mid an^3 + c \\ d \equiv -b \pmod{n}}} 1 = 0 \quad \text{and} \quad \sum_{m=1}^{\infty} \sum_{\substack{d \mid am^3 + c \\ d \equiv -b \pmod{m}}} 1 \leq 3^8 M^{128},$$

where n is any integer with $n > 3^4 M^{66}$.

2. Preliminaries

Let (x, y, z) be any positive integral solution of equation (1). In this section, we shall prove some lemmas which are useful to prove the main results.

Lemma 1. *If $\gcd(c, x) = 1$, then $\gcd(b, x) = 1$.*

Proof. If $\gcd(b, x) = d$ for some integer d , then, by equation (1), we see that $d \mid c$ and hence $d \mid \gcd(x, c) = 1$. This proves the lemma. ■

Lemma 2. Let $\gcd(x, c) = d$. Then we get the positive integers $x_1 = x/d$, $y_1 = \gcd(b, d)y/d$ and $z_1 = zd/\gcd(b, d)$ with $\gcd(x_1, c/d) = \gcd(ad^2, c/d) = 1$, such that $(X, Y, Z) = (x_1, y_1, z_1)$ satisfies the equation

$$ad^2X^3 + \frac{b}{\gcd(b, d)}Y + \frac{c}{d} = XYZ. \quad (2)$$

Proof. Let $d = \gcd(x, c)$. Then by letting

$$x_1 = \frac{x}{d} \quad \text{and} \quad c_1 = \frac{c}{d},$$

from (1), we get,

$$ax_1x^2 + \frac{by}{d} + c_1 = x_1yz.$$

Therefore

$$\frac{by}{d} = x_1yz - ax_1x^2 - c_1.$$

Since $x_1yz - ax_1x^2 - c_1$ is an integer, by/d is a positive integer. Therefore $d \mid by$. This implies that

$$\frac{d}{\gcd(b, d)} \mid y.$$

Let $d_1 = \gcd(b, d)$ and $y_1 = \gcd(b, d)y/d$. So, the tuple (x_1, y_1, z_1) satisfies

$$ad^2x_1^3 + \frac{b}{d_1}y_1 + c_1 = x_1y_1z_1,$$

where $z_1 = zd/\gcd(b, d)$. Since $\gcd(a, c) = 1$ and c is square-free, we have $\gcd(ad^2, c_1) = 1$ and c_1 is square-free. ■

In the above lemma, if we include the condition $c \mid b$, then we have the following result. This gives the converse part also.

Lemma 3. Consider equation (1) with $c \mid b$. Let $\gcd(x, c) = d$. Then we get the positive integers x_1, y and z with $\gcd(ad^2, c/d) = \gcd(x_1, c/d) = 1$, such that $(X, Y, Z) = (x_1, y, z)$ satisfy the equation

$$ad^2X^3 + \frac{b}{d}Y + \frac{c}{d} = XYZ. \quad (3)$$

Conversely, if (x, y, z) is a positive integral solution of equation (3), for some divisor d of c such that $\gcd(x, c/d) = 1$, then (dx, y, z) is a positive solution of equation (1).

Remark 1. Lemma 2 suggests that we can always take a positive solution (x, y, z) of equation (1) with $\gcd(x, c) = 1$. In this case, if the solution (x, y, z) satisfies $x \leq f_1(a, b, c)$, $y \leq f_2(a, b, c)$ and $z \leq f_3(a, b, c)$ for some polynomial functions f_i 's in a, b and c with positive coefficients, then, in the general case, the solution (x', y', z') of equation (1) satisfies $x' \leq cf_1(ac^2, b, c)$, $y' \leq cf_2(ac^2, b, c)$ and $z' \leq cf_3(ac^2, b, c)$.

Remark 2. Since $y|(ax^3 + c)$ and $(xz - b)|(ax^3 + c)$, an upper bound of x gives immediately upper bounds for y and z via $y \leq ax^3 + c$ and $z \leq ax^2 + c + b$.

Lemma 4. Assume that $\gcd(x, c) = 1$. Then there exist positive integers l and r with $xl = by + c$, such that $(X, Y, Z) = (l, y, r)$ satisfy the equation

$$cX^3 + abc^2Y + ac^3 = XYZ. \quad (4)$$

Proof. Since $\gcd(x, c) = 1$, by Lemma 1, we have $\gcd(x, b) = 1$. As $ax^3 + by + c = xyz$, we see that $x \mid (by + c)$ and $y \mid (ax^3 + c)$. Therefore, let $l = (by + c)/x$. Then $y \mid (xl - c)$. As, $y \mid (ax^3 + c)$, we have $y \mid (cl^3 + ac^3)$. Therefore $y \mid (cl^3 + abc^2y + ac^3)$. Also, as $l \mid (by + c)$, we conclude that $l \mid (cl^3 + abc^2y + ac^3)$.

Let $\lambda = \gcd(l, y)$. Then, as $y \mid (xl - c)$, we have $\lambda \mid c$ and hence $\lambda \mid \gcd(y, c)$. Since $\gcd(x, c) = 1$ and $\lambda \mid \gcd(y, c)$, we get $\lambda \mid a$. Hence $\lambda \mid \gcd(a, c) = 1$. Therefore $\gcd(l, y) = 1$. Then there exists a positive integer r such that

$$cl^3 + abc^2y + ac^3 = lyr.$$

This proves the lemma. ■

Lemma 5. Assume that $\gcd(x, c) = 1$. Then there exists a positive integral solution $(l(x), y, l(z))$ of equation (4) satisfying the following;

- (i) $xl(x) = by + c$.
- (ii) If $cl(x) \geq x$, then $l(z) > b$.
- (iii) If $ax \geq l(x)$, then $z > b$.
- (iv) If $x \geq (ac + 2)/(2a - 1)$ and $l(x) \geq x + 2$, then $z \leq ab$.
- (v) If $l(x) \geq c + 2$ and $x \geq l(x) + 2$, then $l(z) \leq abc^2$.

Proof. By Lemma 4, there exist positive integers $l(x)$ and $l(z)$ such that

$$cl(x)^3 + abc^2y + ac^3 = l(x)yl(z)$$

and

$$xl(x) = by + c.$$

This proves (i).

Since $xl(x) = by + c$ and $x \leq cl(x)$, we have $c \leq cl(x)^2 - by$. Suppose that $l(z) \leq b$. Then we get,

$$c \leq cl(x)^2 - by \leq cl(x)^2 - l(z)y = -(abc^2y + ac^3)/l(x) < 0,$$

which is a contradiction. This proves (ii).

Since $xl(x) = by + c$ and $ax \geq l(x)$, we have $c \leq ax^2 - by$. Suppose that $z \leq b$. Then we get

$$c \leq ax^2 - by \leq ax^2 - zy = -(by + c)/x < 0,$$

which is a contradiction. This proves (iii).

Now, we put $y = (xl(x) - c)/b$ in $ax^3 + by + c = xyz$. Then we get,

$$z = \left(\frac{ax^2 + l(x)}{xl(x) - c} \right) b.$$

Therefore, to prove (iv), it is enough to prove that if

$$x \geq \frac{ac + 2}{2a - 1} \quad \text{and} \quad l(x) \geq x + 2,$$

then,

$$\left(\frac{ax^2 + l(x)}{xl(x) - c} \right) \leq a.$$

Suppose that $(ax^2 + l(x))/(xl(x) - c) > a$. Then $l(x) < (ax^2 + ac)/(ax - 1)$. Since $x \geq (ac + 2)/(2a - 1)$, we have $(ax^2 + ac)/(ax - 1) \leq x + 2$. Hence, $l(x) < x + 2$ which is a contradiction. Therefore,

$$\left(\frac{ax^2 + l(x)}{xl(x) - c} \right) \leq a.$$

Now, we shall assume that

$$l(x) \geq c + 2 \text{ and } x \geq l(x) + 2.$$

We prove that $l(z) \leq abc^2$. Putting $by = l(x)x - c$ in

$$cl(x)^3 + abc^2y + ac^3 = l(x)yl(z),$$

we get,

$$l(z) = \left(\frac{l(x)^2 + acx}{xl(x) - c} \right) bc.$$

Therefore, to prove (v), it is enough to prove that

$$\left(\frac{l(x)^2 + acx}{xl(x) - c} \right) \leq ac.$$

Assume that $(l(x)^2 + acx)/(xl(x) - c) > ac$. Then, we get,

$$x < (l(x)^2 + c)/(l(x) - 1).$$

Since $x \geq l(x) + 2$, we get

$$l(x) + 2 < (l(x)^2 + c)/(l(x) - 1)$$

and hence

$$l(x) < c + 2,$$

a contradiction. Hence (v) follows. This proves the lemma. ■

Lemma 6. *For any non-zero integers x , a and c , we have*

$$\gcd(ax^2 + x - 1, x^2 - x - c) \quad \text{divides} \quad |a^2c^2 - 3ac - a - c|$$

and

$$\gcd(ax^2 + x + 1, x^2 + x - c) \quad \text{divides} \quad |a^2c^2 + 3ac + a - c|.$$

Proof. Let $d = \gcd(ax^2 + x - 1, x^2 - x - c)$. Then $d \mid (ax^2 + x - 1)$ and $d \mid (x^2 - x - c)$. It is clear that if $q \mid A$ and $q \mid B$ for any integers q , A and B , then $q \mid A - B$. From this argument, we have the first assertion. To get the second assertion, replace x by $-x$ and a by $-a$ in the first assertion and get the result. ■

3. Proof of Theorem 1

Proof. Let (x, y, z) be any positive integral solution of equation (1). By Remark 1, it is enough to assume that $\gcd(x, c) = 1$. Therefore, by Lemma 1, we have $\gcd(x, b) = 1$.

Case 1: $z \leq ab$. Since $ax^3 + c = (xz - b)y$, we get $(xz - b) \mid (ax^3 + c)$. Since

$$z^3(ax^3 + c) = (xz - b)(az^2x^2 + abxz + ab^2) + (cz^3 + ab^3),$$

we see that $(xz - b) \mid (cz^3 + ab^3)$. Therefore

$$(xz - b) \leq (cz^3 + ab^3).$$

From this, we observe that

$$x \leq cz^2 + ab^3 + b \leq ab^3 + ca^2b^2 + b. \quad (5)$$

and

$$y \leq a(ab^3 + ca^2b^2 + b)^3 + c. \quad (6)$$

Case 2: $z > ab$, $l(x) \geq c + 2$ and $x \geq l(x) + 2$. Then, by Lemma 5, we have $l(z) \leq abc^2$. Therefore, by replacing a by c , b by abc^2 and c by ac^3 in Case 1, we get,

$$y < c(c(abc^2)^3 + ac^5(abc^2)^2 + abc^2)^3 + ac^3.$$

Thus, we get,

$$y \leq ac^3[a^2b^3c^4(a^2b^2c^5 + a^2bc^7 + 1)^3 + 1]. \quad (7)$$

Since $xl(x) = by + c$, we get,

$$x \leq abc^3[a^2b^3c^4(a^2b^2c^5 + a^2bc^7 + 1)^3 + 1] + c. \quad (8)$$

Therefore, by Remark 2, we get

$$z \leq a \{ abc^3[a^2b^3c^4(a^2b^2c^5 + a^2bc^7 + 1)^3 + 1] + c \}^2 + b + c. \quad (9)$$

Case 3: $z > ab$ and, $l(x) < c + 2$ or $x < l(x) + 2$. Suppose that

$$l(x) < c + 2.$$

Then, by Remark 2, we have

$$y \leq c(c + 2)^3 + ac^3.$$

Since $x \leq by + c$, we have

$$x \leq b[c(c + 2)^3 + ac^3] + c.$$

Next we shall assume that $x < l(x) + 2$. By Lemma 4, there exists a positive integral solution $(l(x), y, l(z))$ of equation (4) satisfying $xl(x) = by + c$. Since $z > ab$, by Lemma 5, we conclude that either

$$x < \frac{ac + 2}{2a - 1} \quad \text{or} \quad l(x) < x + 2.$$

Consider the case $l(x) - 2 < x < l(x) + 2$. Suppose that $x = l(x)$. Since $xl(x) = by + c$, we get $x^2 = by + c$. Hence, $y = (x^2 - c)/b$. Put this y in $ax^3 + by + c = xyz$. Then we have

$$z = \frac{bx(ax + 1)}{x^2 - c}.$$

Since $\gcd(x, c) = 1$, $\gcd(x, x^2 - c) = 1$. Hence $x^2 - c \leq b(ax + 1)$. That is, $x(x - ab) \leq b + c$. If $x > ab$, then $x \leq b + c$. Otherwise $x \leq ab$. Hence, $x \leq \max\{b + c, ab\}$.

Suppose that $x = l(x) + 1$. Since $xl(x) = by + c$, $by + c = x(x - 1)$ and so $y = (x^2 - x - c)/b$ and putting this value in equation (1), we get

$$z = \frac{b(ax^2 + x - 1)}{x^2 - x - c}. \quad (10)$$

By Lemma 6, we see that

$$\gcd(ax^2 + x - 1, x^2 - x - c) \quad \text{divides} \quad |a^2c^2 - 3ac - a - c|.$$

Therefore, by equation (10), we get,

$$x^2 - x - c \leq b|a^2c^2 - 3ac - a - c|.$$

Thus, we get,

$$x^2 - x - c \leq |a^2bc^2 - 3abc - ab - cb|.$$

Hence, we arrive at,

$$x \leq |a^2bc^2 - 3abc - ab - cb| + c.$$

Suppose that $l(x) = x + 1$. Since $xl(x) = by + c$, $by + c = x(x + 1)$ and so $y = (x^2 + x - c)/b$. Put this value of y in equation (1), we get

$$z = \frac{b(ax^2 + x + 1)}{x^2 + x - c}.$$

By Lemma 6, we get,

$$x^2 + x - c \leq b|a^2c^2 + 3ac + a - c|.$$

Therefore, we get

$$x \leq [b|a^2c^2 + 3ac + a - c| + c]^{1/2}.$$

By Remarks 1 and 2, and equations (10), (11) and (12), we get the bounds. This proves the theorem. $\blacksquare$

4. Proof of Theorem 2

Let n be any positive integer. Let d be a positive divisor of $an^3 + c$ such that $d \equiv -b \pmod{n}$. Then there exists a positive integer m such that $d = mn - b$. Since $(mn - b) \mid (an^3 + c)$, there is a positive integer y such that $an^3 + c = (mn - b)y$ which in turn satisfies $an^3 + by + c = myn$. That is, for a positive divisor d of $an^3 + c$ with $d \equiv -b \pmod{n}$, we get a positive integral solution (n, y, m) of (1). Indeed, for any two distinct positive divisors d_1 and d_2 , $d_1 \equiv -b \pmod{n}$ and $d_2 \equiv -b \pmod{n}$, of $an^3 + c$, we get distinct positive integral solutions of (1). Therefore, we get,

$$\sum_{n=1}^{\infty} \sum_{\substack{d \mid an^3 + c \\ d \equiv -b \pmod{n}}} 1 \leq N(a, b, c).$$

For the other inequality, let (n, y, z) be a positive integral solution of (1). Then we see that $(nz - b)$ divides $an^3 + c$ and $nz - b$ is positive as y and $an^3 + c$ are positive. By letting $d = nz - b$, we get a positive divisor of $an^3 + c$ which is $\equiv -b \pmod{n}$. Thus, we get,

$$N(a, b, c) \leq \sum_{n=1}^{\infty} \sum_{\substack{d \mid an^3 + c \\ d \equiv -b \pmod{n}}} 1.$$

These inequalities prove the theorem.

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