A SUFFICIENT CONDITION FOR $(\theta^N)_N$ TO HAVE A DISTRIBUTION MODULO ONE, WHEN θ IS IN $\mathbb{F}_2(X)$

J-M. Deshouillers
Université de Bordeaux, Bordeaux INP et CNRS, UMR 5251, 33405 TALENCE Cedex, France
jean-marc.deshouillers@math.u-bordeaux.fr

R. Thangadurai
Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahbad, 211019, India
thanga@hri.res.in

Received: , Revised: , Accepted: , Published:

Abstract

Let θ be a given element in $\mathbb{F}_2(X)$. In this article, we give a sufficient condition for the sequence $(\theta^n)_{n\geq 0}$ to have a distribution modulo 1.

1. Introduction

Many number theoretic problems have natural counterparts in the domain of function fields. We are concerned here with the question of the distribution modulo 1 of the powers of an element $\theta \in \mathbb{F}_q(X)$, the counterpart of the question of the distribution modulo 1 of $(3/2)^n$. The reader will notice that the method and result of this note can easily be extended to the case of an algebraic element over $\mathbb{F}_q(X)$; since our result is only partial, we see no interest in stating it in a more general form, as long as generalisation does not bring a better understanding.

Let us start by giving some definition. We denote $\mathbb{F}_q((X))$ by the set of all the Laurent expansions

$$\eta = \sum_{k\geq -k_0} \varepsilon_k(\eta)X^k, \quad k_0 \in \mathbb{N} \text{ and } \varepsilon_k(\eta) \in \mathbb{F}_q.$$

It is a field which contains $\mathbb{F}_q(X)$.

Definition 1 (Densities)**.** *Let $\theta \in \mathbb{F}_q((X))$. We say that the sequence $(\theta^n)_{n\geq 0}$ has a distribution modulo 1 if for any $L \geq 1$ and for any $b_L \in \mathbb{F}_q^L$, the sequence*

$$\mathcal{N}(\theta, b_L) = \{n \in \mathbb{N} \colon (\varepsilon_1(\theta^n), \ldots, \varepsilon_L(\theta^n)) = b_L\} \tag{1}$$

has an asymptotic density, i.e., if the following limit

$$\lim_{x\to\infty} \frac{1}{x} \operatorname{Card}\{n \leq x \colon n \in \mathcal{N}(\theta, b_L)\} \tag{2}$$

exists.

Similarly, we say that the sequence $(\theta^n)_{n\geq 0}$ has a logarithmic distribution modulo 1 if for any $L \geq 1$ and for any $b_L \in \mathbb{F}_q^L$, the sequence $\mathcal{N}(\theta, b_L)$ has a logarithmic density, i.e., if the following limit

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\in\mathcal{N}(\theta,b_L), n\leq x} \frac{1}{n} \tag{3}$$

exists.

Houndonougbo proved in [5] the existence of the distribution modulo 1 of the sequence $(\theta^n)_{n\geq 0}$, where $\theta = P(X)^\mu + 1/P(X)^\nu$ for positive integers μ and ν and P a non constant polynomial in $\mathbb{F}_q[X]$: he indeed showed more, namely that the sequence $\mathcal{N}(\theta, (0, 0, \ldots, 0))$ has density 1. Deshouillers proved in [4] that the sequence $(\theta^n)_{n\geq 0}$ also has a distribution modulo 1 when $\theta = P(X)/X^\nu$, *i.e.* when the Laurent expansion of θ is finite: he showed that for any b_L the sequence $\mathcal{N}(\theta, b_L)$ is q-automatic and that it has a density. Allouche and Deshouillers proved in [1] that for any θ algebraic over $\mathbb{F}_q(X)$, the sequence $\mathcal{N}(\theta, b_L)$ is q-automatic; by a general result of Cobham [3], this implies that the sequence $(\theta^n)_{n\geq 0}$ has a logarithmic distribution modulo 1, but the existence of a distribution modulo 1 is still an open question.

Our aim is to provide a criterion which is sufficient to prove the existence of the distribution modulo 1 of $(\theta^n)_{n\geq 0}$. We made some ten hand numerical experiments on θ with an infinite Laurent expansion; in the cases we considered, this criterion turned out to be satisfied and indeed led to a limit distribution which is the Dirac measure at 0.

From now on, we assume that $q = 2$ and that $\theta \in \mathbb{F}_2(X)$. In order to describe the 2-automata which generate the sequences $\mathcal{N}(\theta, b_L)$ we follow [1] and first introduce some definition.

For $n \geq 0$, we consider the Laurent expansions

$$\theta^n = \sum_{k \geq -k_0(n)} \varepsilon_k(\theta^n) X^k.$$

Since θ is rational, its expansion is ultimately periodic and the following definition makes sense.

Definition 2 (Parameter)**.** *The* parameter *of an element θ in $\mathbb{F}_2(X)$ is the smallest even positive integer T satisfying*

$$\varepsilon_{-h}(\theta) = 0 \text{ if } h \geq T$$

and

$$\varepsilon_{h+T}(\theta) = \varepsilon_h(\theta) \text{ if } h \geq T.$$

From now on we denote T as the parameter of θ. For $n \geq 0$ and $K, L \geq 0$, we define

$$\mathcal{B}(n, K, L) = (\varepsilon_{-K}(\theta^n), \varepsilon_{-K+1}(\theta^n), \ldots, \varepsilon_{L-1}(\theta^n), \varepsilon_L(\theta^n)) \in \mathbb{F}_2^{L+1+T}, \tag{4}$$

$$\mathcal{B}(n, L) = \mathcal{B}(n, T, L) \tag{5}$$

and

$$\mathcal{M}(n) = (m(n, 0), \ldots, m(n, T-1)) \in \mathbb{F}_2^T, \tag{6}$$

$$\text{where, for } t \in \mathbb{Z}\colon m(n, t) = \sum_{h=0}^{\infty} \varepsilon_{-t-hT}(\theta^n) \in \mathbb{F}_2,$$

which is well defined since this sum contains only a finite number of non-zero elements.

The key ingredient in [1] is the fact that, for $L \geq T$, the two $(2T + L + 1)$-tuples $(\mathcal{M}(2n), \mathcal{B}(2n, L))$ and $(\mathcal{M}(2n + 1), \mathcal{B}(2n + 1, L))$ only depend on $(\mathcal{M}(n), \mathcal{B}(n, L))$. Since [1] is not easily available, we give here a proof of this fact.

Proposition 1. *Let $L \geq T$; there exist two maps ρ and τ from $\mathbb{F}_2^{2T+L+1}$ into itself such that for every $n \geq 0$ one has,*

$$(\mathcal{M}(2n), \mathcal{B}(2n, L)) = \rho\left((\mathcal{M}(n), \mathcal{B}(n, L))\right), \tag{7}$$

$$(\mathcal{M}(2n + 1), \mathcal{B}(2n + 1, L)) = \tau\left((\mathcal{M}(n), \mathcal{B}(n, L))\right). \tag{8}$$

Proof. We first observe that

$$\forall k \in \mathbb{Z}\colon \varepsilon_{2k}(\theta^{2n}) = \varepsilon_k(\theta^n), \tag{9}$$

$$\forall k \in \mathbb{Z}\colon \varepsilon_{2k+1}(\theta^{2n}) = 0, \tag{10}$$

$$\text{For } t \text{ even in } [0, T)\colon m(2n, t) = m(n, t/2) + m(n, t/2 + T/2), \tag{11}$$

$$\text{For } t \text{ odd in } [0, T)\colon m(2n, t) = 0. \tag{12}$$

This implies that as soon as one knows $\mathcal{B}(n, L)$, all the coefficients of θ^{2n} with indices beween $-2T-1$ and $2L+1$ are known: so are $\mathcal{B}(2n, 2T+1, 2L+1)$ and *a fortiori* $\mathcal{B}(2n, L)$. Similarly, the knowledge of $\mathcal{M}(n)$ implies that of $\mathcal{M}(2n)$. This implies (7).

We noticed that the knowledge of $\mathcal{B}(n, L)$ gives us that of $\mathcal{B}(2n, 2T+1, 2L+1)$. Let us show that the knowledge of $\mathcal{B}(n, L)$ and of $\mathcal{M}(n)$ gives us the knowledge of

$$m(2n, t) = \sum_{h=0}^{\infty} \varepsilon_{-t-hT}(\theta^{2n}) \text{ for } t \in [-2L-T-1, 3T+1]. \tag{13}$$

Indeed, if $t \in [0, T-1]$, then $m(2n, t)$ is an element of $\mathcal{M}(2n)$; Otherwise $m(2n, t)$ is an element of $\mathcal{M}(2n)$ which is modified by a few terms which belong to $\mathcal{B}(2n, 2T+1, 2L+T+1)$, *e.g.* $m(2n, -2) = \varepsilon_2(\theta^{2n}) + m(2n, T-2)$, $m(2n, T) = -\varepsilon_0(\theta^{2n}) + m(2n, 0)$.

For any k we have

$$\begin{aligned}
\varepsilon_k(\theta^{2n+1}) &= \sum_{r=-\infty}^{+\infty} \varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n}) = \sum_{r=0}^{+\infty} \varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n}) \\
&= \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n}) + \sum_{r=2T}^{\infty} \varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n}) \\
&= \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n}) + \sum_{\nu=0}^{T-1} \varepsilon_{T+\nu}(\theta) \sum_{\substack{r \geq 2T \\ r \equiv \nu \bmod T}} \varepsilon_{k+T-r}(\theta^{2n}) \\
&= \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta)\varepsilon_{k+T-r}(\theta^{2n}) + \sum_{\nu=0}^{T-1} \varepsilon_{T+\nu}(\theta)\, m(2n, T+\nu-k).
\end{aligned}$$

The last relation shows that as soon as one knows $\mathcal{B}(n, L)$ and $\mathcal{M}(n, L)$ (and the digits of θ with indices between $-T$ and $2T$ which are our inititial data), we have enough information to determine $\mathcal{B}(2n+1, L)$ (cf. (13) and the fact that for $k \in [-T, L]$ we have $T+\nu-k \in [T-L, 3T-1] \subset [-2L-T-1, 3T+1]$).

We finally study $\mathcal{M}(2n+1)$. Let $t \in [0, T-1)$. Reasoning as above, we have.

$$\begin{aligned}
m(2n+1, t) &= \sum_{h=0}^{\infty} \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta)\varepsilon_{-t-r-(h-1)T}(\theta^{2n}) \\
&+ \sum_{h=0}^{\infty} \sum_{r=2T}^{\infty} \varepsilon_{-T+r}(\theta)\varepsilon_{-t-r-(h-1)T}(\theta^{2n}) \\
&= S_1 + S_2, \quad \text{say.}
\end{aligned}$$

By interchanging the sums in the first term on the right hand side, we see that it is equal to

$$S_1 = \sum_{r=0}^{2T-1} \varepsilon_{-T+r}(\theta)\, m(2n, t+r-T). \tag{14}$$

Since $r \in [0, 2T-1]$ and $t \in [0, T-1]$, we have $-T \le t+r-T \le 2T-2$ and thus the term in (14) is known as soon as $\mathcal{M}(n)$ is known. Let us look at the second term. We have

$$\begin{aligned} S_2 &= \sum_{h=0}^{\infty}\sum_{r=2T}^{\infty} \varepsilon_{-T+r}(\theta)\varepsilon_{-t-r-(h-1)T}(\theta^{2n}) \\ &= \sum_{h=0}^{\infty}\sum_{s=T}^{\infty} \varepsilon_{s}(\theta)\varepsilon_{-t-s-hT}(\theta^{2n}) \\ &= \sum_{h=0}^{\infty}\sum_{\nu=0}^{T-1} \sum_{\substack{r\ge T \\ s\equiv \nu \bmod T}} \varepsilon_{s}(\theta)\varepsilon_{-t-s-hT}(\theta^{2n}). \end{aligned}$$

We use the periodicity of the digits of θ and write $s = \nu + T + kT$. We have

$$\begin{aligned} S_2 &= \sum_{\nu=0}^{T-1} \varepsilon_{T+\nu}(\theta) \sum_{h=0}^{\infty}\sum_{k=0}^{\infty} \varepsilon_{-t-\nu-T-(h+k)T}(\theta^{2n}) \\ &= \sum_{\nu=0}^{T-1} \varepsilon_{T+\nu}(\theta) \sum_{\ell=0}^{\infty} \left(\sum_{\substack{h\ge 0, k\ge 0 \\ h+k=\ell}} 1 \right) \varepsilon_{-t-\nu-T-\ell T}(\theta^{2n}). \end{aligned}$$

It is enough to consider each inside sum over ℓ. We notice that if $t+\nu$ is odd, then all the terms $\varepsilon_{-t-\nu-T-\ell T}(\theta^{2n})$ are zero and so is the sum of those terms over ℓ. We also notice that the sum $\sum_{\substack{h\ge 0, k\ge 0 \\ h+k=\ell}} 1$ is equal to 1 when ℓ is even and to 0 when ℓ is odd. Combining those two remarks and writing $\ell = 2\lambda$, we have, when $t+\nu$ is even

$$\begin{aligned} &\sum_{\ell=0}^{\infty} \left(\sum_{\substack{h\ge 0, k\ge 0 \\ h+k=\ell}} 1 \right) \varepsilon_{-t-\nu-T-\ell T}(\theta^{2n}) = \sum_{\lambda=0}^{\infty} \varepsilon_{-(\nu+t+T)/2-\lambda T}(\theta^{n}) \\ = \;& m(n, (\nu+t+T)/2); \end{aligned}$$

when $\nu + t + T \le 2T$, then $m(n, (\nu+t+T)/2)$ is an element in $\mathcal{M}(n)$; otherwise, we write $m(n, (\nu+t+T)/2) = m(n, (\nu+t-T)/2) - \varepsilon_{(T-\nu-t)/2}(\theta^n)$, which the difference of an element of $\mathcal{M}(n)$ and an element of $\mathcal{B}(n, L)$.

Thus S_2 is also known as soon as $\mathcal{B}(n, L)$ and $\mathcal{M}(n)$ are known. This ends the proof of Proposition 1. $\square$

This permits to build a directed graph Γ_L with edges indexed by 0 or 1 as follows. We first consider the set of vertices

$$\mathcal{R}_L = \{(\mathcal{M}(n), \mathcal{B}(n, L)) : n \geq 0\}. \tag{15}$$

We then build two edges starting from $r_L \in \mathcal{R}_L$ depending on $\varepsilon = 0$ or $\varepsilon = 1$ in the following way: since $r_L = (\mathcal{M}(n), \mathcal{B}(n, L))$ for some integer $n \geq 0$, for each $\varepsilon \in \{0, 1\}$, we define the state

$$\delta_L(r_L, \varepsilon) := (\mathcal{M}(2n + \varepsilon), \mathcal{B}(2n + \varepsilon, L)) \tag{16}$$

and the edge $(r_L, \delta_L(r_L, \varepsilon))$, which are well defined. The above-mentioned observation of [1] implies that our definition is indeed independent of the choice of n such that $r_L = (\mathcal{M}(n), \mathcal{B}(n, L))$. The reader who needs to refresh her/his knowledge on *Automatic Sequences* is strongly recommended to visit [2][1] , specially subsections 4.1 and 5.1. The refreshment being performed, it is not difficult to see that the sequence $\mathcal{N}(\theta, b_L)$ is 2-automatic: it is recognized by the automaton $\mathcal{A}(b_L) = \{\mathcal{R}_L, \{0, 1\}, \delta_L, r_{0,L}, F(b_L)\}$, where $\mathcal{R}_L$ and δ_L have already been defined,

$$r_{0,L} = (\mathcal{M}(0), \mathcal{B}(0, L))$$

and

$F(b_L)$ is the set of those $r \in \mathcal{R}_L$, the last L components of which are b_L.

It will be convenient to extend the function δ_L to a new function still called δ_L, defined over the words w on $\{0, 1\}$, satisfying

$$\forall r \in \mathcal{R}_L, \forall \varepsilon \in \{0, 1\}, \forall w \in \{0, 1\}^* : \delta_L(r, \emptyset) = r, \delta_L(r, \varepsilon w) = \delta_L(\delta_L(r, w), \varepsilon).$$

Let us recall a criterion on the graph Γ_L which insures that the sequence $\mathcal{N}(\theta, b_L)$ has a density.

In the directed graph Γ_L, we say that two vertices r and s are *equivalent* if there is a directed path leading from r to s and a directed path leading from s to r; this permits to consider *equivalent classes*, which form a tree, which leads to the notion of *final class*; we finally say that an equivalent class is *regular* if there exists an integer ℓ such that for any pair (r, s), there is a directed path of length ℓ leading from r to s. We have the following criterion.

Proposition 2. *Let $L \geq T$ be a given integer and $b_L \in \mathbb{F}_2^L$ be a given vector. If the graph Γ_L of the automaton $\mathcal{A}(b_L)$ has a single final class and if this class is regular, then, the sequence $\mathcal{N}(\theta, b_L)$ has an asymptotic density.*

[1]Thanks, Jeff, for this invaluable monography... and for the rest!

Theorem 8.4.7, p. 272 of [2] deals with the special case where all the states constitute a single final regular equivalence class. Proposition 2 is a mere extension of this result where the key tool is Perron-Frobenius theorem.

Back to our question, we notice that if $u \leq v$, then $\mathcal{N}(\theta, b_u)$ is a finite union of sequences $\mathcal{N}(\theta, b_v)$, and it is thus enough for our purpose to consider the sequences $\mathcal{N}(\theta, b_L)$ for all sufficiently large values of L.

We wish to prove here that if the criterion applies to an automaton $\mathcal{A}(b_L)$, it also applies to the automaton $\mathcal{A}(b_{L+1})$, which leads to the following

Theorem 1. *Let θ be an element in $\mathbb{F}_2(X)$ and let T be its parameter. If the graph Γ_T has a single final class and if this final class is regular, then the sequence $(\theta^n)_{n\geq 0}$ has a distribution modulo 1.*

We remark that in the statement of Theorem 1, the automaton $\mathcal{A}(b_T)$ only occurs through its graph Γ_T, which itself depends on $\{\mathcal{R}_T, \delta_T, r_{0,T}\}$ but not on $F(b_T)$.

Corollary 1. *Let θ be an element in $\mathbb{F}_2(X)$ and let T be its parameter. If the graph Γ_T has a single equivalence class, then the sequence $(\theta^n)_{n\geq 0}$ has a distribution modulo 1.*

2. Connection between the automata $\mathcal{A}(b_L)$ and $\mathcal{A}(b_{L+1})$

Our key tool to understand the connection between the automata $\mathcal{A}(b_{L+1})$ and $\mathcal{A}(b_L)$ is a natural map from $\mathcal{R}_{L+1}$ onto $\mathcal{R}_L$; we define it and give its main properties in the following proposition.

Proposition 3. *Let θ and T be as in Theorem 1 and let $L \geq T$. The map σ_L from $\mathcal{R}_{L+1}$ to $\mathbb{F}_2^{2T+L+1}$, defined by suppressing the last component of an element, has the following properties*

$$\sigma_L\left(\mathcal{R}_{L+1}\right) = \mathcal{R}_L, \tag{17}$$

$$\forall r \in \mathcal{R}_L,\ \textit{at most two elements } s \in \mathcal{R}_{L+1} \textit{ such that } \sigma_L(s) = r, \tag{18}$$

$$\sigma_L(r_{0,L+1}) = r_{0,L}, \tag{19}$$

$$\forall r \in \mathcal{R}_{L+1},\ \textit{and } \varepsilon \in \{0,1\},\ \textit{we have} : \sigma_L\left(\delta_{L+1}(r,\varepsilon)\right) = \delta_L(\sigma_L(r),\varepsilon). \tag{20}$$

Proof. By definition, cf. (15), for a state r_{L+1} in $\mathcal{R}_{L+1}$, there exists an integer n such that r_{L+1} is the $(2T+L+2)$-tuple $(\mathcal{M}(n), \mathcal{B}(n, L+1))$. If we suppress its last component we get the $(2T+L+1)$-tuple $(\mathcal{M}(n), \mathcal{B}(n, L))$, which is an element of $\mathcal{R}_L$. In the other direction, if we start with an element r_L in $\mathcal{R}_L$, there exits an n such that $r_L = (\mathcal{M}(n), \mathcal{B}(n, L))$, and for this n we have $r_L = \sigma_L((\mathcal{M}(n), \mathcal{B}(n, L+1))$.

Thus, the suppression of the last component defines the map σ_L which satisfies the Property (17). Property (18) comes from the fact that the last component of an element of $\mathcal{R}_{L+1}$ belongs to $\{0,1\}$. We have $\sigma_L(r_{0,L+1}) = \sigma_L\left((\mathcal{M}(0), \mathcal{B}(0, L+1)\right) = (\mathcal{M}(0), \mathcal{B}(0, L)) = r_{0,L}$, which proves Property (19). Finally, let $r \in \mathcal{R}_{L+1}$, there exists an integer n such that $r = (\mathcal{M}(n), \mathcal{B}(n, L+1))$; for $\varepsilon \in \{0,1\}$, we have, cf. (16), $\delta_{L+1}(r, \varepsilon) := (\mathcal{M}(2n+\varepsilon), \mathcal{B}(2n+\varepsilon, L+1))$ and so $\sigma_L\left(\delta_{L+1}(r,\varepsilon)\right) = \sigma_L\left((\mathcal{M}(2n+\varepsilon), \mathcal{B}(2n+\varepsilon, L+1))\right) = (\mathcal{M}(2n+\varepsilon), \mathcal{B}(2n+\varepsilon, L)) = \delta_L(\sigma_L(r), \varepsilon)$, which proves (20). □

We say that $r_{L+1} \in \mathcal{R}_{L+1}$ is *sitting above* r_L for some $r_L \in \mathcal{R}_L$, if $\sigma(r_{L+1}) = r_L$.

Claim 1. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L and let $\mathcal{C}_{L+1}$ be one of the final classes of Γ_{L+1}. Then any element of $\mathcal{C}_{L+1}$ is sitting above some element of $\mathcal{C}_L$.*

Proof. Let $r_{L+1} \in \mathcal{C}_{L+1}$ be a given element. Then we have, $\sigma_L(r_{L+1}) \in \mathcal{R}_L$. If $\sigma_L(r_{L+1}) \in \mathcal{C}_L$, then we are done. If not, then there exists a word $w \in \{0,1\}^*$ such that $\delta_L(\sigma_L(r_{L+1}), w) \in \mathcal{C}_L$. Since r_{L+1} belongs to $\mathcal{C}_{L+1}$ which is a final class, the state $\delta_{L+1}(r_{L+1}, w)$ belongs to $\mathcal{C}_{L+1}$. Therefore, there exists a word $w' \in \{0,1\}^*$ such that $\delta_{L+1}(r_{L+1}, ww') = r_{L+1}$. Thus, by the definition of $\mathcal{C}_L$, we see that $\delta_L(\sigma_L(r_{L+1}), ww') \in \mathcal{C}_L$. Since r_{L+1} is sitting above $\sigma_L(r_{L+1})$, by (20) we conclude that $r_{L+1} = \delta_{L+1}(r_{L+1}, ww')$ is sitting above $\delta_L(\sigma_L(r_{L+1}), ww') \in \mathcal{C}_L$. □

Now, we look at a converse of Claim 1.

Claim 2. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L and let $\mathcal{C}_{L+1}$ be one of the final classes of Γ_{L+1}. For any element $r \in \mathcal{C}_L$, there is an element $s \in \mathcal{C}_{L+1}$ which is sitting above r.*

Proof. Let r be an element in $\mathcal{C}_L$ and t an element in $\mathcal{C}_{L+1}$. There exists a word $w \in \{0,1\}^*$ such that $\delta_L(\sigma_L(t), w) \in \mathcal{C}_L$. Since $r \in \mathcal{C}_L$, there exists a word w' such that $\delta_L(\sigma_L(t), ww') = r$. Now, we look at $\delta_{L+1}(t, ww')$. Since $t \in \mathcal{C}_{L+1}$, we conclude that $\delta_{L+1}(t, ww') \in \mathcal{C}_{L+1}$. Since t is sitting above $\sigma_L(t)$, by (20), we have $\delta_{L+1}(t, ww') \in \mathcal{C}_{L+1}$ is sitting above r. □

Claim 3. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L. If $\mathcal{C}_{L+1}$ is a final class in Γ_{L+1}, then $|\mathcal{C}_{L+1}| \geq |\mathcal{C}_L|$.*

Proof. If $r \neq s$ are two different elements in $\mathcal{C}_L$, then, by Claim 2, there exist two elements r', s' of $\mathcal{C}_{L+1}$ such that r' is sitting above r and s' is sitting above s. Since $r \neq s$, by the definition, we see that $r' \neq s'$. Hence the claim. □

Claim 4. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L. Then there can be at most two distinct final classes in Γ_{L+1}.*

Proof. We first remark that two classes are distinct if and only if they are disjoint; now, the claim simply follows from Claim 3 and (18): each final class has at least $|\mathcal{C}_L|$ elements and for any element r in $\mathcal{C}_L$ we can have at most two elements of $\mathcal{R}_{L+1}$ sitting above r. $\square$

Claim 5. *Assume that $\mathcal{C}_L$ is the unique final class in Γ_L. Suppose that $\mathcal{C}_{L+1}^{(1)}$ and $\mathcal{C}_{L+1}^{(2)}$ are two distinct final classes in Γ_{L+1} sitting above $\mathcal{C}_L$. They are disjoint and for every element $r \in \mathcal{C}_L$, there is exactly one element $r_1 \in \mathcal{C}_{L+1}^{(1)}$ and one element $r_2 \in \mathcal{C}_{L+1}^{(2)}$ which are sitting above r. Moreover, for any $(r_1, r_2) \in \mathcal{C}_{L+1}^{(1)} \times \mathcal{C}_{L+1}^{(2)}$ and for any word $w \in \{0,1\}^*$, we have*

$$\delta_{L+1}(r_1, w) \neq \delta_{L+1}(r_2, w). \tag{21}$$

Proof. Since the two classes $\mathcal{C}_{L+1}^{(1)}$ and $\mathcal{C}_{L+1}^{(2)}$ are distinct, they are disjoint. By Claim 2, above each element $r \in \mathcal{C}_L$, there exist $r_1 \in \mathcal{C}_{L+1}^{(1)}$ and $r_2 \in \mathcal{C}_{L+1}^{(2)}$ sitting above r ; since the two classes are disjoint, we have $r_1 \neq r_2$. By (18), this implies that above r, there can be only one element from each of the $\mathcal{C}_{L+1}^{(i)}$. The last assertion follows the fact that for any word w, the element $\delta_{L+1}(r_i, w)$ is in $\mathcal{C}_{L+1}^{(i)}$.

$\square$

3. Proof of Theorem 1

By Proposition 2, it is enough to prove that for any $L \geq T$ and any b_L, the graph Γ_L of the automaton $\mathcal{A}(b_L)$ has a single final class, and this class is regular. Let us recall that Γ_L is independent of b_L. We shall prove our assertion by induction on L.

The assumption of Theorem 1 is simply the case $L = T$: the graph Γ_T has a single final class and this class is regular.

Let us assume that for some $L \geq T$, the graph Γ_L has a single final class and this class is regular; let $\mathcal{C}_L$ be this class.

We first prove that the graph Γ_{L+1} has a single final class. By Claim 1, any single final class of Γ_{L+1} is sitting above $\mathcal{C}_L$; thus by Claim 4, there are at most two final classes in Γ_{L+1}. If we have indeed two distinct final classes, then we can apply Claim 5: let r be an element in $\mathcal{C}_L$: there exist two elements r_1 and r_2 which are sitting above r and which belong to the two different classes above $\mathcal{C}_L$. Choose an integer h such that $2^h > L + 1$ and consider a word w consisting of h zeroes. By (20), the elements $\delta_{L+1}(r_i, w)$ are sitting above $\delta_L(r, w)$ for $i = 1$ and 2. This means that they differ at most by their last digit. Let n_1 and n_2 be two integers such that

$$r_i = (\mathcal{M}(n_i), \mathcal{B}(n_i, L+1)) \text{ for } i = 1, 2.$$

Then, we have

$$\begin{aligned}\delta_{L+1}(r_i, w) &= \left(\mathcal{M}\left(n_i 2^h\right), \mathcal{B}\left(n_i 2^h, L+1\right)\right)\\ &= \left(\mathcal{M}\left(n_i 2^h\right), \mathcal{B}\left(n_i 2^h, L\right) \circ \varepsilon_{L+1}(\theta^{n_i 2^h})\right)\end{aligned}$$

for $i = 1, 2$, where the symbol $\circ$ represents the concatenation. Since $2^h > L+1$, we have

$$\varepsilon_{L+1}(\theta^{n_1 2^h}) = \varepsilon_{L+1}(\theta^{n_2 2^h}) = 0,$$

which implies

$$\delta_{L+1}(r_1, w) = \delta_{L+1}(r_2, w),$$

a contradiction to Claim 5. Hence, there is only one final class in the graph Γ_{T+1}.

Let $\mathcal{C}_{L+1}$ be the unique final class in Γ_{L+1}. It remains to show that this class is regular.

If $|\mathcal{C}_{L+1}| = |\mathcal{C}_L|$, then, by Claim 2, for every $r_L \in \mathcal{C}_L$, there is exactly only one $r_{L+1} \in \mathcal{C}_{L+1}$ such that r_{L+1} is sitting above r_L and conversely. Therefore, by (20), $\delta_{L+1}(r_{L+1}, w)$ is sitting above $\delta_L(r_L, w)$ for every word w. Since $\mathcal{C}_L$ is regular, this implies that $\mathcal{C}_{L+1}$ is also regular.

Suppose that $|\mathcal{C}_{L+1}| > |\mathcal{C}_L|$. Then, there exists r in $\mathcal{C}_L$ such that the two elements $r \circ 0$ and $r \circ 1$ are in $\mathcal{C}_{L+1}$. Choose an integer h such that $2^h > L+1$ and let w be the word consisting of h zeroes. By the above argument, we have

$$\delta_{L+1}(r \circ 0, w) = \delta_{L+1}(r \circ 1, w),$$

and we denote this element by s. Since $\mathcal{C}_L$ is regular, there exists an integer K such that for any $k \geq K$, there is a word w_k of length k satisfying

$$\delta_L(\sigma_L(s), w_k) = r;$$

thus $\delta_{L+1}(s, w_k)$ is either $r \circ 0$ or $r \circ 1$. But in either case, we have $\delta_{L+1}(s, w_k w) = s$, so that for any $\ell \geq K + h$ there is a path of length ℓ which connects s to s. Since $\mathcal{C}_{L+1}$ is an equivalent class, any element u can be connected to any element v by a path of length exactly $2|\mathcal{C}_{L+1}| + K + h$, which implies that $\mathcal{C}_{L+1}$ is regular. Theorem 1 is proved.

□

Proof of Corollary 1. Since the graph of $\mathcal{A}(b_T)$ has unique class, say, $\mathcal{C}_T$, it is the final class. Choose an integer h such that $2^h > T$. Let w be a word consisting only h zeroes. Then $\delta_T(r_{0,T}, w) = \left(\mathcal{M}(2^h), \mathcal{B}(2^h, T)\right) = r_h$, say. Therefore there exists a word w' such that $\delta_T(r_h, w') = r_{0,T}$. Thus, we have

$$\delta_T(r_{0,T}, ww') = r_{0,T} \text{ with } |ww'| = K, \text{ say.}$$

Since it is an equivalent class, any element u can be connected to any other element v by a path of length $K + 2|\mathcal{C}_T|$. Hence it is regular. Therefore, by Theorem 1, we get the corollary. $\square$

Acknowledgements. The authors thank CEFIPRA: project 5401-A permitted to complete this work. The first author acknowledges with thanks the support of the *Programme Franco-Indien pour les Mathématiques*, a CNRS LIA programme, and the MuDeRa programme, supported by the Austrian FWF and the French ANR. The second author is thankful to IMB, Université de Bordeaux, hosting institute, for providing necessary support and excellent atmosphere.

Both authors are thankful to the referee for her/his careful reading and suggestions for improving the accessibility of the paper.

References

[1] J.-P. Allouche and J-M. Deshouillers, *Répartition de la suite des puissances dune série formelle algébrique*, in: Colloque de Théorie Analytique des Nombres à la mémoire de Jean Coquet, Journées SMF-CNRS, CIRM Luminy 1985, Publications Mathématiques d'Orsay, 88-02 (1988), 37-47.

[2] J.-P. Allouche and J. Shallit, *Automatic Sequences; Theory, Applications and Generalizations*, Cambridge University Press, Cambridge, (2003).

[3] A. Cobham, On the base-dependence of sets of numbers recognizable by finite automata, *Math. Systems Theory*, **3** (1969), 186-192.

[4] J-M. Deshouillers, *La répartition modulo* 1 *des puissances d'un élément dans* $\mathbb{F}_q((X))$. (French) [Distribution modulo 1 of the powers of an element in $\mathbb{F}_q((X))$], Recent progress in analytic number theory, Vol. 2 (Durham, 1979), pp. 69-72, Academic Press, London-New York, 1981.

[5] V. Houndonougbo, Mesure de répartition d'une suite $(\theta^n)_{n\in\mathbb{N}}$ dans un corps de séries formelles sur un corps fini. (French), *C. R. Acad. Sci. Paris Sr. A*, **288** (1979), no. 22, 997-999.