Fine's theorem, noncontextuality, and correlations in Specker's scenario

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Operational theories and Ontological models

An operational theory is specified by a triple, $(\mathcal{P}, \mathcal{M}, p)$, where $\mathcal{P}$ denotes the preparation procedures $P \in \mathcal{P}$ in the lab, $\mathcal{M}$ denotes the measurement procedures $(M, \mathcal{K}_M) \in \mathcal{M}$, and $p : (\mathcal{P}, \mathcal{M}) \to [0, 1]$ is the probability $p(k|P, M)$ that measurement outcome $k \in \mathcal{K}_M$ is observed when measurement procedure $(M, \mathcal{K}_M)$ is implemented following the preparation procedure P.

Operational theories and Ontological models

An ontological model (Λ, Ξ, μ) of an operational theory $(\mathcal{P}, \mathcal{M}, p)$ posits a space of ontic states $\lambda \in \Lambda$, probability densities $\mu : \Lambda \to [0, \infty)$ defined over which constitute the preparation procedures, and response functions $\xi : (\Lambda, \mathcal{M}) \to [0, 1]$ denote the probability $\xi(k|M, \lambda)$ that measurement outcome $k \in \mathcal{K}_M$ is observed when measurement procedure $(M, \mathcal{K}_M)$ is implemented and the ontic state of the system is λ.

Operational theories and Ontological models

An ontological model of an operational theory must be empirically adequate, that is:

$$p(k|P, M) = \int d\lambda \mu_P(\lambda)\xi(k|M, \lambda), \tag{1}$$

for all $P \in \mathcal{P}, (M, \mathcal{K}_M) \in \mathcal{M}$. This is how an operational theory and its ontological model fit together.

Operational theories and Ontological models

- So far we haven't imposed any restriction on either the operational theory or its ontological model. We are interested in the question of whether an operational theory still admits an ontological model if an additional feature is required of such a model: noncontextuality.
- Also, the operational theory will be assumed to be quantum theory for the purpose of this talk.

Noncontextuality[1]

A methodological principle: If two experimental procedures are operationally indistinguishable then they must also be ontologically indistinguishable—the ontological identity of operational indiscernables.

[1] R. W. Spekkens, Phys. Rev. A 71, 052108 (2005)

Noncontextuality

The experimental procedures we are interested in are preparations and measurements. An ontological model of an operational theory is noncontextual if it satisfies two properties: *preparation noncontextuality* and *measurement noncontextuality*.

Noncontextuality

Preparation noncontextuality (PNC) is expressed as the following inference from the operational theory to its ontological model:

$$\begin{aligned} & p(k|P,M) = p(k|P',M), \forall (M,\mathcal{K}_M) \in \mathcal{M} \text{ or } P \simeq P' \\ \Rightarrow \quad & \mu_P(\lambda) = \mu_{P'}(\lambda), \forall \lambda \in \Lambda. \end{aligned} \tag{2}$$

That is, two preparations P and P' which are operationally indistinguishable are represented by identical distributions in the ontological model. Note that $P \simeq P'$ denotes the operational equivalence of preparation procedures P and P'.

Noncontextuality

Measurement noncontextuality (MNC) is expressed as the following inference:

$$
\begin{aligned}
& p(k|P, M) = p(k|P, M'), \forall P \in \mathcal{P} \text{ or } M \simeq M' \\
\Rightarrow \quad & \xi(k|M, \lambda) = \xi(k|M', \lambda), \forall \lambda \in \Lambda. \qquad (3)
\end{aligned}
$$

That is, measurements M and M' which do not differ in their statistics relative to all preparations $P \in \mathcal{P}$ are represented by identical response functions in the ontological model. Note that $M \simeq M'$ denotes the operational equivalence of measurement procedures M and M'.

Outcome determinism (OD)

Outcome determinism (OD) is the assumption that the ontological response functions for every measurement procedure $(M, \mathcal{K}_M) \in \mathcal{M}$ in the operational theory are deterministic: $\xi(k|M, \lambda) \in \{0, 1\}$ for all $k \in \mathcal{K}_M, \lambda \in \Lambda$.

When the operational theory is quantum mechanics...

1. PNC $\Rightarrow$ ODSM, where ODSM is outcome determinism for all sharp (projective) measurements.
2. MNC and ODSM together constitute the assumption of KS-noncontextuality.
3. However, ODUM, or outcome determinism for all unsharp (nonprojective) measurements is an additional assumption that does not follow from noncontextuality.
4. In applying MNC to unsharp measurements, therefore, we do not want to assume ODUM. There exist proofs of contextuality assuming ODUM[2] but as argued by Spekkens[3], assuming ODUM with MNC allows a trivial contradiction using only the fair coin flip POVM, $\{\frac{I}{2}, \frac{I}{2}\}$.
5. We want to test the possibility of a noncontextual model of quantum mechanics without assuming ODUM.

[2] e.g., A. Cabello, PRL 90, 190401 (2003); P. Busch, PRL 91, 120403 (2003)

[3] R.W. Spekkens, Found. Phys. 44, 1125 (2014); see also A. Grudka and P. Kurzynski, PRL 100, 160401 (2008)

Fine's theorem[4]

- As originally stated, Fine's theorem referred to the Bell-CHSH scenario and showed (among other things) that given experimental correlations in this scenario, a locally deterministic model exists if and only if a locally causal model exists. This is a compelling argument for why outcome determinism is not an assumption required to prove Bell's theorem.
- Naively, one might expect this to be true of the KS theorem as well, especially when it comes to what are called "state-dependent" proofs that rely on violating KS inequalities. However, Fine's theorem turns out to be of limited utility in noncontextual models, especially when the full class of outcome-indeterministic models is allowed.

[4] A. Fine, PRL 48, 291 (1982)

Theorem

Given a set of measurements $\{M_1, \ldots, M_N\}$ *with jointly measurable subsets* $S \subset \{1, \ldots, N\}$*, where each measurement* $M_s, s \in S$*, takes values labelled by* $k_s \in \mathcal{K}_{M_s}$*, the following propositions are equivalent:*

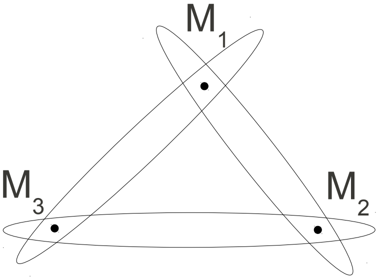
1. *For a given preparation* $P \in \mathcal{P}$ *of the system there exists a joint probability distribution* $p(k_1, \ldots, k_N|P)$ *that recovers the marginal statistics for jointly measurable subsets predicted by the operational theory (such as quantum theory) under consideration, i.e.,* $\forall S \subset \{1, \ldots, N\}$*,* $p(k_S|M_S; P) = \sum_{k_i : i \notin S} p(k_1, \ldots, k_N|P)$*, where* $k_S \in \mathcal{K}_{M_S}$*.*
2. *There exists a measurement-noncontextual and outcome-deterministic, i.e. KS-noncontextual, model for these measurements.*
3. *There exists a measurement-noncontextual and factorizable model for these measurements.*

Does Fine's theorem rescue ODUM? No.

- It is often left implicit, though sometimes explicity suggested[5], that "ruling out all possible noncontextual deterministic hidden variable models implies ruling out all possible noncontextual stochastic models as well."
- This is not strictly true: only factorizable hidden variable models are excluded by any argument of the KS-type. Factorizable models are those which require factorizability of joint measurement response functions, e.g., $\xi(X_1, X_2|M_{12}, \lambda) = \xi(X_1|M_1, \lambda)\xi(X_2|M_2, \lambda)$.
- There exist nonfactorizable hidden variable models that aren't ruled out by KS-type arguments. To rule out these, we need to go beyond KS inequalities.
- LSW inequality and its variants are noncontextuality inequalities that address precisely this question.

[5] C. Simon, C. Brukner, A. Zeilinger, PRL 86, 4427 (2001)

Specker's scenario



Three binary-outcome measurements, denoted M_1, M_2, and M_3, each with outcome space $\{0, 1\}$, for which every pair is jointly measurable.

Specker's scenario

For every pair $\{M_i, M_j\}$ where $(i,j) \in \{(1,2),(2,3),(1,3)\}$, there is a four-outcome measurement M_{ij}, such that the operational statistics of measurements M_i and M_j are recovered as marginals of the operational statistics of M_{ij}. We denote the outcome of M_{ij} by (X_i, X_j) and let $M_i^{(j)}$ ($M_j^{(i)}$) denote the coarse-graining over X_j (X_i) of M_{ij}:

$$p(X_i|M_i^{(j)}, P) \equiv \sum_{X_j} p(X_i, X_j|M_{ij}, P), \tag{4}$$

$$p(X_j|M_j^{(i)}, P) \equiv \sum_{X_i} p(X_i, X_j|M_{ij}, P). \tag{5}$$

Specker polytope

For any preparation, the data in this scenario consists of 12 probabilities, four from each pairwise distribution $p(X_i, X_j | M_{ij}, P)$. We can express the assumption of pairwise joint measurability of M_1, M_2 and M_3 as the following operational equivalences

$$M_1^{(2)} \simeq M_1^{(3)} \simeq M_1, \tag{6}$$

$$M_2^{(1)} \simeq M_2^{(3)} \simeq M_2, \tag{7}$$

$$M_3^{(1)} \simeq M_3^{(2)} \simeq M_3. \tag{8}$$

Along with positivity and normalization, this condition (often called *no disturbance*) defines a polytope in $\mathbb{R}^6$ with 12 vertices, 8 deterministic and 4 indeterministic.

The deterministic vertices are obvious: they are the 8 extremal KS-noncontextual assignments that can be made to outcomes of M_1, M_2, M_3. The convex hull of these vertices defines the KS-noncontextuality polytope for this scenario; its halfspace representation is given by the following set of four KS inequalities:

$$R_3 \equiv p(X_1 \neq X_2|M_{12}, P)+p(X_2 \neq X_3|M_{23}, P)+p(X_1 \neq X_3|M_{13}, P) \leq 2,$$

$$R_0 \equiv p(X_1 \neq X_2|M_{12}, P)-p(X_2 \neq X_3|M_{23}, P)-p(X_1 \neq X_3|M_{13}, P) \leq 0,$$

$$R_1 \equiv p(X_2 \neq X_3|M_{23}, P)-p(X_1 \neq X_2|M_{12}, P)-p(X_1 \neq X_3|M_{13}, P) \leq 0,$$

$$R_2 \equiv p(X_1 \neq X_3|M_{13}, P)-p(X_1 \neq X_2|M_{12}, P)-p(X_2 \neq X_3|M_{23}, P) \leq 0.$$

The indeterministic vertices are the following:

$$\forall(ij) : p(X_i = 0, X_j = 1|M_{ij}, P) = p(X_i = 1, X_j = 0|M_{ij}, P) = \frac{1}{2}, \tag{9}$$

and the other three for $(kl) \in \{(12), (23), (13)\}$:

$$\begin{aligned} & p(X_k = 0, X_l = 1|M_{kl}, P) = p(X_k = 1, X_l = 0|M_{kl}, P) = \frac{1}{2}, \\ & \forall(ij) \neq (kl) : \\ & p(X_i = 0, X_j = 0|M_{ij}, P) = p(X_i = 1, X_j = 1|M_{ij}, P) = \frac{1}{2}, \end{aligned} \tag{10}$$

Each indeterministic vertex violates exactly one corresponding KS inequality maximally.

LSW and its three variants: the measurements

- The LSW inequality and its variants bound correlations outside the KS-noncontextuality polytope but still within the larger Specker polytope by taking into account the effect of noise in qubit measurements.
- We take $\{M_1, M_2, M_3\}$ to be three qubit measurements which are pairwise jointly measurable: M_i is associated with the qubit POVM $\{E_0^{(i)}, E_1^{(i)}\}$, where $E_b^{(i)}$ corresponds to the outcome $X_i = b$.

In particular, we assume

$$E_0^{(i)} \equiv \frac{1}{2}I + \frac{1}{2}\eta_0\vec{\sigma}\cdot\hat{n}_i, \quad E_1^{(i)} \equiv \frac{1}{2}I - \frac{1}{2}\eta_0\vec{\sigma}\cdot\hat{n}_i,$$

where $\vec{\sigma} \equiv (\sigma_x, \sigma_y, \sigma_z)$ is the vector of qubit Pauli matrices, and I is the identity matrix. Note that $E_b^{(i)} = \eta_0 \Pi_b^{(i)} + (1-\eta_0)\frac{I}{2}$, where $\Pi_b^{(i)} = \frac{1}{2}I + (-1)^b \frac{1}{2}\vec{\sigma}\cdot\hat{n}_i$.

Predictability

We define the *predictability* of measurement M_i to be:

$$\eta_{M_i} \equiv \max_{P \in \mathcal{P}} \{2 \max\{p(X_i = 0|M_i, P), p(X_i = 1|M_i, P)\} - 1\},$$

We have $\eta_{M_i} = 1$ for perfect predictability - there exists some preparation which makes M_i yield a deterministic outcome - and $\eta_{M_i} = 0$ for perfect unpredictability - there exists no preparation that can make the outcome of M_i anything other than uniformly random. The quantum measurements we have chosen yield $\eta_{M_i} = \eta_0$ for all $i \in \{1, 2, 3\}$.

LSW and its three variants: the inequalities

$$
\begin{aligned}
R_3 &\leq 3 - \eta_0, \\
R_0 &\leq 1 - \eta_0, \quad (11) \\
R_1 &\leq 1 - \eta_0, \quad (12) \\
R_2 &\leq 1 - \eta_0. \quad (13)
\end{aligned}
$$

Note than for $\eta_0 = 1$, these reduce to the KS inequalities. However, pairwise joint measurability of qubit POVMs requires $\eta_0 < 1$: this is why the KS inequalities do not suffice to account for noncontextuality in this scenario. In quantum theory, there do not exist sharp measurements which are pairwise jointly measurable and still allow a violation of KS inequalities.

Equivalence under relabelling

- The three inequalities on R_0, R_1, and R_2 are equivalent to the LSW inequality under the following relabellings of measurement outcomes: $X_3 \to X_3' \equiv 1 - X_3$ (takes R_3 to R_0), $X_2 \to X_2' \equiv 1 - X_2$ (takes R_3 to R_2), $X_1 \to X_1' \equiv 1 - X_1$ (takes R_3 to R_1).
- For example: $X_3 \to X_3' \equiv 1 - X_3$ means $p(X_2 \neq X_3'|M_{23}, P) = 1 - p(X_2 \neq X_3|M_{23}, P)$, $p(X_1 \neq X_3'|M_{13}, P) = 1 - p(X_1 \neq X_3|M_{13}, P)$, and therefore:

$$\begin{aligned} R_3' &= p(X_1 \neq X_2|M_{12}, P) + p(X_2 \neq X_3'|M_{23}, P) \\ &+ p(X_1 \neq X_3'|M_{13}, P) \leq 3 - \eta_0 \end{aligned}$$

is equivalent to $R_0 \leq 1 - \eta_0$.

The anticorrelation vs. predictability tradeoff

- The LSW inequality quantifies the tradeoff between the achievable anticorrelation and the achievable predictability for a set of measurements. Noncontextuality implies that one cannot achieve both to an arbitrary degree: $R_3 + \eta_0 \leq 3$. This inequality is nontrivial for all $\eta_0 > 0$.
- Observing perfect anticorrelation ($R_3 = 3$) should not be surprising unless one has also verified nonzero predictability ($\eta_0 > 0$) in the measurement outcomes.

Aside: nonfactorizability in response functions

The single measurement response functions are given by

$$\xi(X_i|M_i;\lambda) = \eta\delta_{X_i,X_i(\lambda)} + (1-\eta)\left(\frac{1}{2}\delta_{X_i,0} + \frac{1}{2}\delta_{X_i,1}\right), \tag{14}$$

$i \in \{1,2,3\}$, in keeping with the assumption of ODSM but not ODUM. The pairwise response function maximizing anticorrelation:

$$\xi(X_i,X_j|M_{ij};\lambda) = \eta\delta_{X_i,X_i(\lambda)}\delta_{X_j,X_j(\lambda)} + (1-\eta)\left(\frac{1}{2}\delta_{X_i,0}\delta_{X_j,1} + \frac{1}{2}\delta_{X_i,1}\delta_{X_j,0}\right). \tag{15}$$

The pairwise response function minimizing anticorrelation:

$$\xi(X_i,X_j|M_{ij};\lambda) = \eta\delta_{X_i,X_i(\lambda)}\delta_{X_j,X_j(\lambda)} + (1-\eta)\left(\frac{1}{2}\delta_{X_i,0}\delta_{X_j,0} + \frac{1}{2}\delta_{X_i,1}\delta_{X_j,1}\right). \tag{16}$$

Quantum violation

On account of the equivalence to LSW inequality under relabelling, all four noncontextuality inequalities admit quantum violation for an appropriate choice of states and measurements [6]. Unlike earlier attempts assuming ODUM, this violation is nontrivial precisely because it works without an assumption of ODUM.

[6] R. Kunjwal and S. Ghosh, Phys. Rev. A 89, 042118 (2014)

Takeaway

- Outcome determinism is an assumption of the KS theorem. It cannot be done away with by appealing to Fine's theorem: all that Fine's theorem excludes are factorizable noncontextual models, which form a restricted class in the space of outcome-indeterministic models.
- The *no disturbance* correlations in Specker's scenario form a polytope in $\mathbb{R}^6$ with 12 vertices, 8 deterministic and 4 indeterministic. Each indeterminstic vertex violates the corresponding KS inequality and noncontextuality inequality (if nontrivial, i.e., $\eta_0 > 0$) maximally.
- The big open question is: how does one extend this analysis to the case where the operational theory is not assumed to be quantum theory?

References

1. Y. C. Liang, R. W. Spekkens, and H. M. Wiseman, Phys. Rep. 506, 1 (2011).
2. R. Kunjwal and S. Ghosh, Phys. Rev. A 89, 042118 (2014)
3. R. W. Spekkens, Found. Phys. 44, 1125 (2014).
4. R. Kunjwal, Phys. Rev. A 91, 022108 (2015).
5. Perimeter Institute seminar (2014), PIRSA:14010102, http://pirsa.org/14010102.

Thanks!