Witnessing genuine multipartite entanglement with positive maps

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Based on the following papers..

Marcus Huber and Ritabrata Sengupta. Witnessing genuine multipartite entanglement with positive maps. *Phys. Rev. Lett*, 113, 100501.

Otfried Gühne, Marcus Huber, Cécilia Lancien, and Ritabrata Sengupta. *On semidefinite relaxations of the separable states*, arXiv:1503.*****.

1. The problem
   - Quantum states and entanglement
   - Role of positive map
2. The multipartite case

Quantum states

A state is a positive semi-definite Hermitian operator with unit trace in $\mathcal{B}(\mathcal{H})$ where (in our case) $\mathcal{H} = \mathbb{C}^n$. If the state is of unit rank, i.e. it is a projection operator, then it is called as pure state. Corresponding is also represented by the corresponding normalised vector $|\psi\rangle \in \mathcal{H}$.

In a bipartite quantum system $H_A \otimes H_B$, a state $\rho_{AB} \in \mathcal{B}(H_A \otimes H_B)$ is said to be separable if it is possible to decompose it as follows:

$$\rho_{AB} = \sum_{j=1}^{m} p_j \rho_j^A \otimes \rho_j^B; \qquad p_j \geq 0, \ \sum_j p_j = 1;$$

where ρ_j^A and ρ_j^B are states in systems A and B respectively.

A bipartite state $\rho \in \mathcal{B}(H_A \otimes H_B)$ is said to be entangled (i.e. not separable) if it can not be written in the above form.

Problem

Given a state in a bipartite system, declare whether it is separable or not.

However for the mixed states, the problem is not so easy. In face it is NP hard.

Existence of witness

Theorem (Horodecki 1997)

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces. For any inseparable state $\rho \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$, there exists a Hermitian operator A such that $Tr(A\rho) < 0$ but $Tr(A\sigma) \geq 0$ for all separable state σ.

This is showing the existence of a hyperplane to separate them.

Any bipartite state $\rho \in \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$, is separable if and only if for all positive maps $\phi : \mathcal{B}(\mathcal{H}_2) \longrightarrow \mathcal{B}(\mathcal{H}_1)$, $\mathbb{1}_{\mathcal{H}_1} \otimes h(\rho)$ is positive.

Definitions

A Hermiticity preserving map $\phi : \mathcal{B}(\mathbb{C}^m) \longrightarrow \mathcal{B}(\mathbb{C}^n)$ is said to be a positive map (P) if for any positive semi-definite Hermitian operator $A \in \mathcal{B}(\mathbb{C}^m)$, $\phi(A) \in \mathcal{B}(\mathbb{C}^n)$ is also positive semi-definite.

ϕ is said to be k-positive, if the map $\mathbb{1}_k \otimes \phi : \mathcal{B}(\mathbb{C}^k \otimes \mathbb{C}^m) \longrightarrow \mathcal{B}(\mathbb{C}^k \otimes \mathbb{C}^n)$ is also a positive map.

ϕ is said to be completely positive (CP) if for all $k = 2, 3, \cdots$ $\mathbb{1}_k \otimes \phi$ are positive maps.

Kraus representation

Any CP map ϕ can be represented by a finite set of matrices $\{V_i \in \mathcal{M}_{n\times m}\}_{i=1}^s$ such that $\phi(A) = \sum_{i=1}^s V_i A V_i^\dagger$.

Throughout this talk, we are going to consider finite dimensional cases only.

On partial transpose

Theorem (Peres 1996)

Partial transpose is a necessary condition for separability.

Using earlier results of Jamiołkowski (1972), Horodeckis' (1997) showed that

Positivity under Partial transpose (PPT) is necessary and sufficient condition for all separable states in $2 \otimes 2,\ 2 \otimes 3$ and $3 \otimes 2$ systems; but is only necessary for higher dimensions (as well as multipartite systems).

In all other dimensions there are examples of states which are PPT.

We call any map which can be written as $CP + T \circ CP$ as decomposable. Clearly these can not detect PPT entangled states. Such examples can be found in $3 \otimes 3$, $2 \otimes 4$, $2 \otimes 2 \otimes 2$ and all subsequent higher dimensional systems.

Choi's map

Lin. Alg. & App. 12(2), 95–100 (1975)

$$\phi : \mathcal{B}(\mathbb{C}^3) \to \mathcal{B}(\mathbb{C}^3)$$
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \to \frac{1}{2}\begin{pmatrix} a_{11}+a_{22} & -a_{12} & -a_{13} \\ -a_{21} & a_{22}+a_{33} & -a_{23} \\ -a_{31} & -a_{32} & a_{33}+a_{11} \end{pmatrix}$$

The above map is 1-positive, 2-positive, not 3-positive and hence not completely positive and *not-decomposable*.

In particular it can detect PPT entangled states. (Choi 1980, Størmer 1982)

There are various generalisations of the above map. Mostly from Seung-Hyeok Kye, and from Dariusz Chruściński and their groups.

Definitions

Entanglement in multipartite systems is complicated. There exists several forms of entanglement for the such systems.

A state is considered to be partially separable with respect to bi-partitions $b \in \mathcal{P}$ if and only if it can be written as

$$\rho_{\mathcal{P}} = \sum_{b \in \mathcal{P}} p_b \left(\sum_i q_b^i (|\phi_i\rangle\langle\phi_i|)_b \otimes (|\phi_i'\rangle\langle\phi_i'|)_{\overline{b}} \right) .$$

The complement of the set of bi-separable states is usually referred to as *genuinely multipartite entangled* states as their creation via LOCC requires pure states that are not separable with respect to any partition.

Given a set of witnesses for different bipartition $\{W_b\}$, i.e. self-adjoint operators with the property that $\mathrm{Tr}(W_b \sigma_b) \geq 0$ we are looking for an operator with the following property

$$W_{GME} \geq W_b \qquad \forall b \in \mathcal{P}$$

Assuming a common operator Q for every bi-partition witness W_b, such that

$$W_b = Q + T_b \,, \tag{1}$$

one can write down a formal solution of the above problem as

$$W_{GME} = Q + \sum_b [T_b]_+ \,, \tag{2}$$

where $[T_b]_+$ denotes the projection onto the positive eigenspace of T_b. It is now easy to see that condition holds for every witness W_b.

Theorem

For any set of bipartite entanglement witness W_b across bi-partitions $b|\overline{b} \in \mathcal{P}$, the following expression is always positive for mixed states ρ, which can be decomposed into pure states that are separable with respect to any of the partitions in $\mathcal{P}$

$$Tr\left[\rho\left(\sum_{b\in\mathcal{P}} \tau_b + Q\right)\right] \geq 0.$$

We have used the abbreviated notation $Q = N + P$ where

$$P = \sum_{\eta,\eta'} |\eta\rangle\langle\eta'| \max[0, \min_{b\in\mathcal{P}}[\Re e[W_b]],$$

$$N = \sum_{\eta,\eta'} |\eta\rangle\langle\eta'| \min[0, \max_{b\in\mathcal{P}}[\Re e[W_b]]$$

and $\tau_b = [W_b - Q]_+$.

Proof

The first observation required is

$$\mathrm{Tr}[\rho[A]_+] \geq \mathrm{Tr}[\rho A]\,, \tag{3}$$

and thus

$$\mathrm{Tr}[\rho(\tau_b + Q)] \geq \mathrm{Tr}[\rho W_b] \tag{4}$$

Now if we write down a state that is decomposable into states ρ_b, separable with respect to bipartition in $\mathcal{B}$ as

$$\rho_{\mathcal{P}} = \sum_b p_b \rho_b \tag{5}$$

we find that

$$\mathrm{Tr}\left[\rho\left(\sum_{b\in\mathcal{P}} \tau_b + Q\right)\right] = \sum_{b\in\mathcal{P}} p_b \mathrm{Tr}[\rho_b(\tau_b + Q)] + \sum_{\{b'\neq b\}\in\mathcal{P}} p'_b \mathrm{Tr}[\rho_{b'}(\tau_b)] \geq 0$$

which completes the proof.

Map connection

Given a multi-partite state ρ than is detected to be entangled across every bipartition by means of positive maps Λ_b. For each bipartition this implies that we will find witnesses of the form

$$W_b(\Lambda_b) = \Lambda_b^* \otimes \mathbb{1}_{\overline{b}}[|\psi_b\rangle\langle\psi_b|].$$

While in the bipartite case the choice of $|\psi_b\rangle\langle\psi_b|$ is obviously given via the eigenvector corresponding to smallest (i.e. negative) eigenvalue of $\Lambda[\rho]$, this choice is not as obvious in the multipartite case. Indeed here we want to minimize the eigenvalues of

$$W_{\neg\mathcal{P}}(\{\Lambda_b, |\psi_b\rangle\}) = \sum_{b\in\mathcal{P}} [\Lambda_b^* \otimes \mathbb{1}_{\overline{b}}[|\psi_b\rangle\langle\psi_b|] - Q]_+ + Q$$

Theorem

For every multi-partite entangled state ρ_{GME} there exist a set of weakly optimal bipartite entanglement witnesses $\{W_b\}$ such that

$$Tr\left[\rho_{GME}\left(Q - \sum_b [T_b]_+\right)\right] < 0$$

Proof: Let Q be a weakly optimal witness for multipartite entanglement, i.e. $\mathrm{Tr}(Q\rho_{bisep}) \geq 0$ and $\exists\rho_b$ s.t. $\mathrm{Tr}(Q\rho_b) = 0$. It is sufficient to show that such a witness can always be constructed from bipartite weakly optimal witnesses as then clearly every multipartite entangled state can be detected by the construction.

$$\max_{\rho_{b'}} \mathrm{Tr}(Q\rho_{b'}) =: \alpha_{b'} \geq 0 \,\forall\, b'\,. \tag{6}$$

As this maximization is convex in the space of states it is clear that the maximal overlap with Q can also be reached by an optimal pure state $|\psi_{b'}\rangle\langle\psi_{b'}|$. Now we can choose the following bipartite witness

contd.

$$W_{b'} = Q - \alpha_{b'} |\psi_b\rangle\langle\psi_b| \,, \tag{7}$$

and verify that indeed $\mathrm{Tr}(W_{b'}\rho_{b'}) \geq 0 \,\forall\, \rho_{b'}$ and $\mathrm{Tr}(W_{b'}|\psi_{b'}\rangle\langle\psi_{b'}|) = 0$, i.e. that this operator is a weakly optimal witness for the bipartition $b'|\overline{b'}$. Inserting this set of optimal bipartite witnesses into the construction from above using $[-\alpha_{b'}|\psi_b\rangle\langle\psi_b|]_+ = 0$ yields

$$W_{GME} = Q \tag{8}$$

proving that each weakly optimal multipartite witness can be gained from the construction using weakly optimal bipartite witnesses.

We can construct examples of states detected by the new witness.

$$|x, y, \lambda_\alpha\rangle = \prod_{i \in \alpha} \sigma_i \otimes \mathbb{1}_{\bar{i}} (\sqrt{\lambda_\alpha}|x\rangle^{\otimes n} + \sqrt{\lambda_\alpha^{-1}}|y\rangle^{\otimes n}) \,,$$

with $\sigma = |x\rangle\langle y| + |y\rangle\langle x|$. And the corresponding projector we denote as $P_\alpha(x, y, \lambda_\alpha)$, such that we can introduce the operator

$$E(\vec{\lambda_\alpha}) = 3|GHZ_3\rangle\langle GHZ_3| + \sum_{i=1,2,3} \sum_{x<y} \sum_{y=1,2} P_i(x, y, \lambda_i) \,,$$

where $|GHZ_3\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)$ and with this finally the density operator

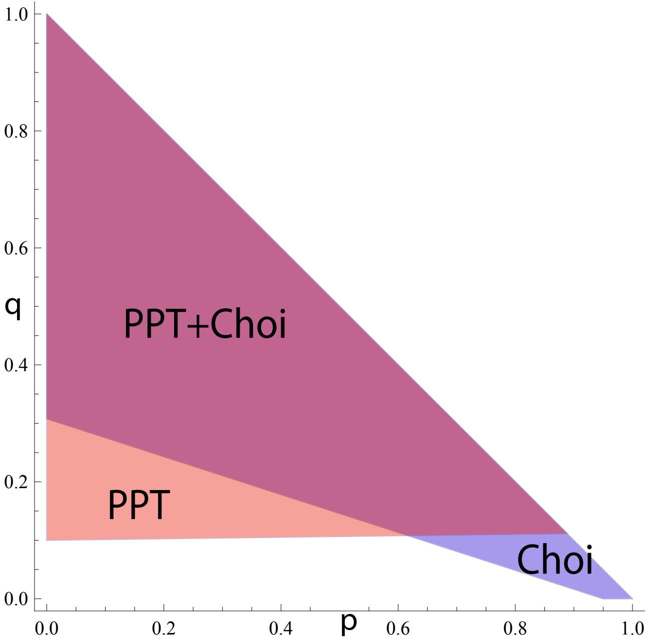
$$\rho(\vec{\lambda_\alpha}) = \frac{E(\vec{\lambda_\alpha})}{\mathrm{Tr}(E(\vec{\lambda_\alpha}))} \,.$$

Take $x, y = 0, 1, 2$ we get a symmetric state we a 27×27 density operator. Putting

$$\rho_{\text{noice}}(p) = p\frac{1}{27}I + (1-p)\rho\left(\frac{1}{9}\right)$$

we find that the white noise resistance, i.e., the critical value of white noise admixture p, until which genuine multipartite entanglement can still be detected is $p_{\text{crit}} = \frac{9}{179}$.
Further we can see the for the two parameter family

$$\rho_{\text{example}} = p\,|GHZ_3\rangle\,\langle GHZ_3| + q\rho\left(\frac{1}{9}\right) + \frac{1-p-q}{27}I.$$


1.0
0.8
0.6
q
0.4
0.2
0.0
PPT+Choi
PPT
Choi
0.0
0.2
0.4
p
0.6
0.8
1.0

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Thank you!!!

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