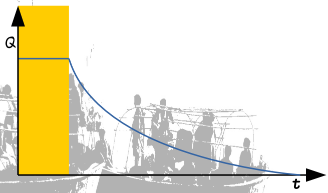
Frozen Quantum Correlations



Amit Kumar Pal

Harish-Chandra Research Institute, Allahabad, INDIA

Young Quantum - 2015

February 24, 2015

Outline

- Introduction
    - Quantum discord & "discord-type" measures
    - Dynamics under local noisy channels
- What is freezing?
    - Initial state, freezing conditions, phase diagram
    - Complementarity, state space, etc.
- Effective freezing
    - Quantitative approach: Freezing index
    - Quantum phase transition using freezing index
- Conclusion

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  3. Several geometric measures

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Review by Modi et. al., RMP (2012)

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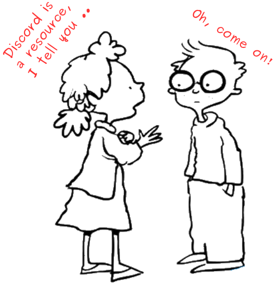

- Even if it is, it faces a problem! **Decoherence**

  Decay of quantum correlations in noisy environments

Measure: Quantum discord

Classical mutual information: Two **equivalent** definitions

- $I(A:B) = H(A) + H(B) - H(A,B)$
- $J(A:B) = H(B) - H(B|A)$

$$H(A) = -\sum_a p_a \log_2 p_a \leftarrow \textbf{Shannon entropy}$$
$$H(B|A) = H(A,B) - H(A) = -\sum_{a,b} p_{a,b} p_{b|a} \leftarrow \textbf{Conditional entropy}$$
$$p_{b|a} = \frac{p_{a,b}}{p_a}$$

Quantum mutual information: Two **inequivalent** definitions

- $\mathcal{I}(A:B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$
- $\mathcal{J}(A:B) = S(\rho_B) - S(\rho_B|\rho_A)$

$$S(\rho) = -\mathrm{Tr}\left[\rho \log_2 \rho\right] \leftarrow \textbf{von Neumann entropy}$$
$$S(\rho_B|\rho_A) = \sum_k p_k S\left(\rho_{AB}^k\right) \leftarrow \textbf{Quantum conditional entropy}$$

$$Q_A(\rho_{AB}) = \min_{\{\Pi_k^A\}} \left\{ S(\rho_A) - S(\rho_{AB}) + \sum_k p_k S\left(\rho_{AB}^k\right) \right\}$$

Minimization over a complete set of projective measurements $\{\Pi_k^A\}$

Ollivier & Zurek, PRL (2001); Henderson & Vedral, J. Phys. A (2001)

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- **Evolution of the state ρ_{AB}:** Kraus operator representation

$$\rho_{AB}(\gamma) = \sum_{\mu,\nu=0}^{1} E_{\mu,\nu}\rho_{AB}(0)E^{\dagger}_{\mu,\nu}; \quad E_{\mu,\nu} = E_{\mu} \otimes E_{\nu}$$

$\gamma \rightarrow$ Decoherence parameter
$0 \leq \gamma \leq 1$

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- **3 types of channels:**

$$E_0 = \sqrt{\frac{1-\gamma}{2}} I_2; \;\; E_1 = \sqrt{\frac{\gamma}{2}}\sigma_{\alpha}$$

  ✓ bit-flip ($\alpha = x$)
  ✓ bit-phase-flip ($\alpha = y$)
  ✓ phase-flip ($\alpha = z$)

Examples

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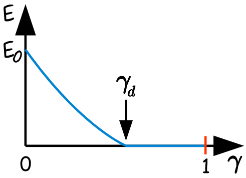

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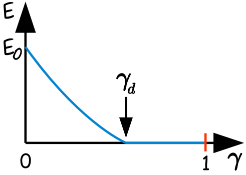

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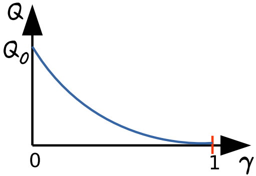

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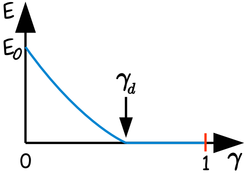

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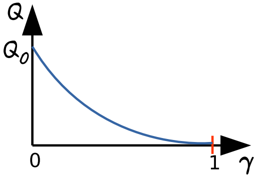

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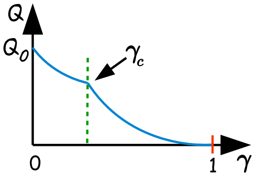

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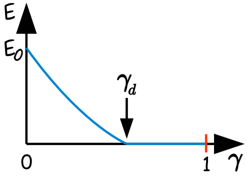

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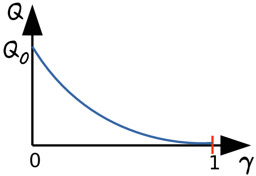

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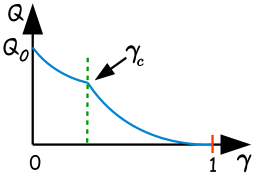

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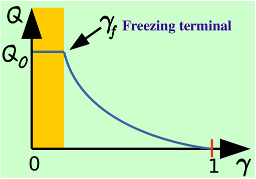

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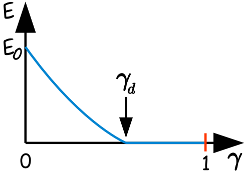

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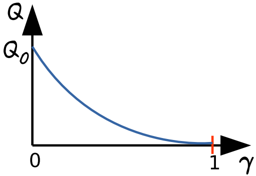

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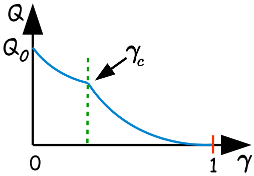

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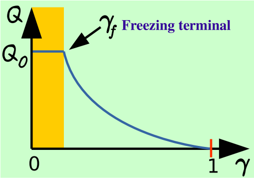

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Adiabatic freezing: Listen to **Debraj's talk!**

Freezing so far..

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- Initial **Bell-diagonal states** under **bit-flip, phase-flip, and bit-phase-flip** evolutions
  - ✓ Quantum discord, quantum work deficit, and several geometric measures
  - ✓ Same initial condition
  - ✓ Experimental realization

Mazzola et. al., PRL (2010)
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- **However, questions remain...**
  1. More general freezing states..?
  2. Conditions for freezing?
  3. Measure-dependent phenomena..?

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$$\rho_{AB} = \frac{1}{4}\left[I_A \otimes I_B + \sum_{\alpha=x,y,z} c_{\alpha\alpha}\sigma_A^\alpha \otimes \sigma_B^\alpha + \sum_{\alpha=x,y,z} c_\alpha^A \sigma_A^\alpha \otimes I_B + \sum_{\beta=x,y,z} c_\beta^B I_A \otimes \sigma_\beta^B \right]$$

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Proposal. Canonical initial state

$$\tilde{\rho}_{AB} = \frac{1}{4}\left[I_A \otimes I_B + \sum_{\alpha=x,y,z} c_{\alpha\alpha}\sigma_A^\alpha \otimes \sigma_B^\alpha + \left(c_x^A \sigma_A^x \otimes I_B + c_x^B I_A \otimes \sigma_B^x \right) \right]$$

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- Under bit-flip evolution, c_{xx}, c_x^A, c_x^B are γ-independent

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- **Freezing of quantum discord with $\tilde{\rho}_{AB}$ as initial state**
  - ✓ Calculate discord of $\tilde{\rho}_{AB}(\gamma)$ as function of γ
  - ✓ Show that discord is γ-independent for a finite interval

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- **Minimization over $\{\Pi_k^A\}$ for general two-qubit state:**
  Minimization over two real paramters, θ $(0 \leq \theta \leq \pi)$, and ϕ $(0 \leq \phi < 2\pi)$

$$\Pi_k^A = U|k\rangle\langle k|U^\dagger; \quad U = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{i\phi} \\ -\sin\frac{\theta}{2}e^{-i\phi} & \cos\frac{\theta}{2} \end{pmatrix}; \quad |k\rangle = |0\rangle, |1\rangle$$

- **Analytical calculation of discord: Bell-diagonal states only**

Luo, PRA (2008)

- **Discord of more general two-qubit mixed state:**
  Despite several attempts, only numerical results so far..

M. Ali et. al., PRA (2010); Lu et. al., PRA (2011);
Girolami & Adesso, PRA (2011); Chen et. al., PRA (2011)

Take help from numerics

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- $\tilde{\rho}_{AB}$ **is special!**

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99.97%

Class A

$\{\Pi_k^A\} \longrightarrow \{\sigma_A^x, \sigma_A^y, \sigma_A^z\}$

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"Special" canonical initial state

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$\{\Pi_k^A\} \longrightarrow$ **Arbitrary**

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Class A	Class B
99.97%	0.03%
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Discord calculated analytically!	Discord calculated numerically!
“Special” canonical initial state	

- **Discord can be calculated analytically for special canonical initial state!**

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> **Assumption:** For 100% of $\tilde{\rho}_{AB}$, $\{\Pi_k^A\} \rightarrow \{\sigma_A^x, \sigma_A^y, \sigma_A^z\}$

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Consequence: $\varepsilon_{abs} = Q_a - Q \leq 2.9088 \times 10^{-3}$

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  1. $(c_{yy}/c_{zz}) = -(c_x^A/c_x^B) = -c_{xx}$, $|c_{xx}| < 1$
  2. $(c_{zz})^2 + (c_x^B)^2 \leq 1$
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  $F(y) = 2\,(H((1+y)/2) - 1)$;
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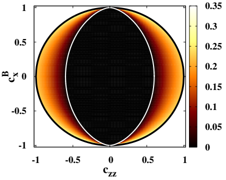
- Valid for all canonical initial states
- γ_f from equality in third condition



Freezing phase diagram
$|c_{xx}| = 0.6$
Different shades: Different values of γ_f

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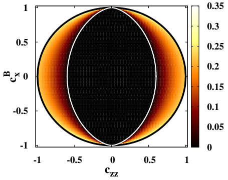

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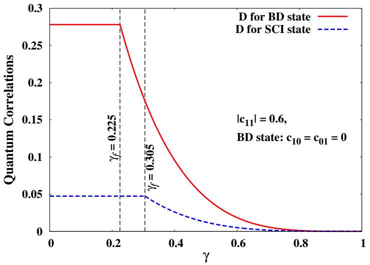

Points to ponder...

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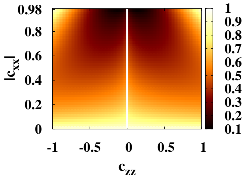
- **Complementarity:** Trade-off between frozen quantum discord and freezing terminal

$$Q_f + \gamma_f \leq 1$$

  ✓ Can be proved for Bell-diagonal state
  ✓ Numerically verified for $\tilde{\rho}_{AB}$



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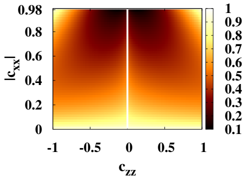

- **Inhomogeneity is crucial!**
$c_x^A = c_x^B \rightarrow |c_{xx}| = 1 \leftarrow$ **Freezing conditions violated**

Points to ponder...

- **Complementarity:** Trade-off between frozen quantum discord and freezing terminal

$$Q_f + \gamma_f \leq 1$$

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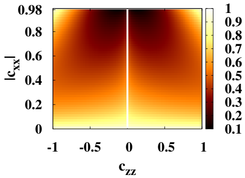

- **Inhomogeneity is crucial!**
  $c_x^A = c_x^B \rightarrow |c_{xx}| = 1 \leftarrow$ **Freezing conditions violated**
- **Freezing states form a non-convex set**

A word on the state space

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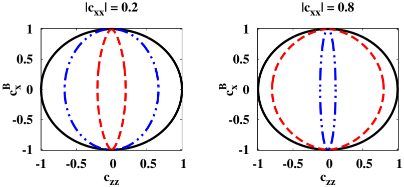
- **Separable states can freeze**



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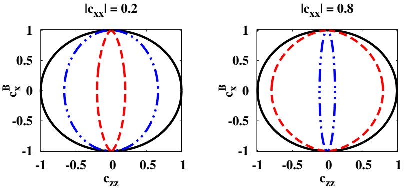

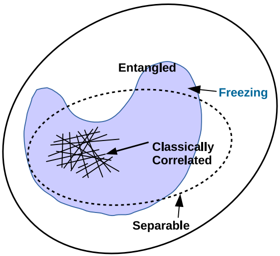

State space: Current picture

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Example: Freezing of **quantum work deficit** of $\tilde{\rho}_{AB}$ under bit-flip evolution

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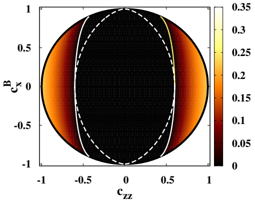

Freezing phase diagram

$|c_{xx}| = 0.6$

Different shades: Different values of γ_f

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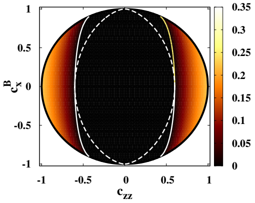

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✓ **Non-convexity** of the set of states $\tilde{\rho}_{AB}$ with freezing quantum work deficit

✓ **Complementarity**

✓ **Inhomogeneity as a crucial freezing condition**

✓ **Freezing of quantum work deficit in separable states**

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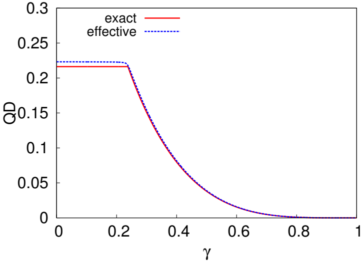

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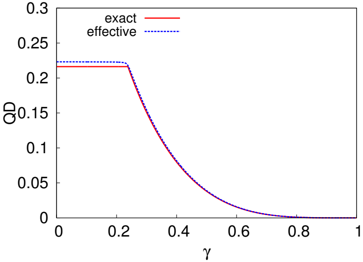

Quantitative description of effective freezing?

How good does a state "almost" freeze?

- **Freezing index:** Quantifies the goodness of freezing

$$\eta_f = \left(\sum_{i=1}^{N_f} \overline{Q}_{f,i} \left(1 - \gamma_{1,i}\right) \int_{\gamma_{1,i}}^{\gamma_{2,i}} Q(\gamma) d\gamma \right)^{\frac{1}{4}}$$

- Depends on
  - ✓ average value of effectively frozen correlation, $\overline{Q}_f$,
  - ✓ duration of freezing, $\Delta\gamma_f$,
  - ✓ onset of $\Delta\gamma_f$, and
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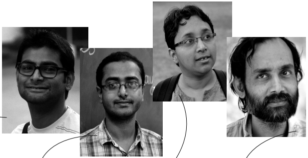
✓ **Detects quantum phase transition** in transverse-field *XY* model

$$H_{XY} = \frac{J}{2} \sum_{i=1}^{N} \left\{ (1+g)\sigma_i^x \sigma_{i+1}^x + (1-g)\sigma_i^y \sigma_{i+1}^y \right\} + h \sum_{i=1}^{L} \sigma_i^z$$

To summarize...

- Freezing in states other than Bell-diagonal state
- Inhomogeneity is crucial
- Choice of quantum correlation measure is important

  $\hookrightarrow$ Different initial conditions
- Complementarity: trade-off between value and duration
- Effective freezing and freezing index



Titas Chanda, Amit Kumar Pal, Anindya Biswas, Aditi Sen(De), and Ujjwal Sen
arXiv: 1409.2096 (2014)

Quantum work deficit

Minimum difference between amount of pure states extractable under suitably restricted "global" and "local" operations

Closed operation (CO)

1. Unitary transformation
2. Dephasing by a set of projectors

$$I_{\mathrm{CO}} = \log_2 \dim(\mathcal{H}) - S(\rho_{AB})$$

Closed local operations & classical communication (CLOCC)

1. Unitary transformation
2. Dephasing by local measurement
3. Sending the dephased qubit via a noiseless quantum channel

$$I_{\mathrm{CLOCC}} = \log_2 \dim(\mathcal{H}) - \min_{\{\Pi_k^A\}} S\left(\rho'_{AB}\right)$$

$$\rho'_{AB} \to \sum_k p_k \rho_{AB}^k$$

$$W_A(\rho_{AB}) = \min_{\{\Pi_k^A\}} \left[S(\sum_k p_k \rho_{AB}^k) - S(\rho_{AB}) \right]$$

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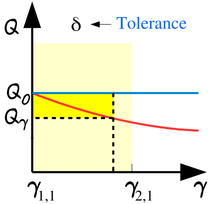
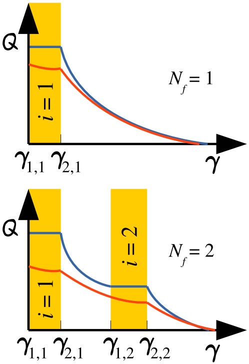


$Q_0 - Q_\gamma < \delta$ for all $\gamma_{1,i} \leq \gamma \leq \gamma_{2,i}$



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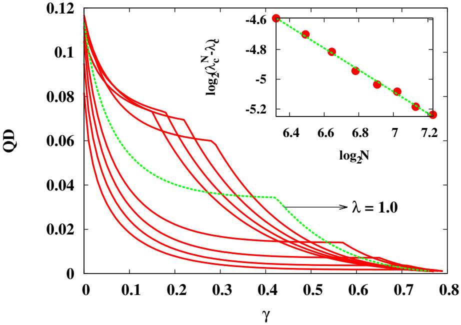
Quantum phase transition with η_f

- *XY* **model in a transverse field**

$$H_{XY} = \frac{J}{2}\sum_{i=1}^{N}\left\{(1+g)\sigma_i^x\sigma_{i+1}^x + (1-g)\sigma_i^y\sigma_{i+1}^y\right\} + h\sum_{i=1}^{L}\sigma_i^z$$

✓ Second order quantum phase transition at $\lambda = h/J = 1$

Lieb, Schultz & Mattis, Ann. Phys. (1961);
Pfeuty, Ann. Phys. (1966); Barouch & McCoy, PRA (1971)



- **Scaling:**
How fast does the system approach thermodynamic limit?

$$\lambda_c^N = \lambda_c + kN^{-0.729}$$