Possibility of adiabatic transport of a Majorana edge state through an extended gapless region

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Introduction to Majorana fermions

1-dimensional p-wave superconductor: phase diagram and topological phases

Adiabatic dynamics and Kibble-Zurek scaling

Violation of Kibble-Zurek scaling for edge Majorana state

Application of complex hopping term

Adiabatic quenching dynamics of periodic chain

Slow quenching dynamics of edge Majorana and possibility of adiabatic transport

Summary
Introduction to Majorana fermions

- A Majorana fermion is a fermion which is its own anti-particle \( c^\dagger = c \)
- Wave function of a Majorana fermion must be real
- The second quantized operators of Majorana fermions satisfy the relations
  \[ c_j^2 = 1 \text{ and } \{ c_j, c_l \} = 2\delta_{jl} \]
- One ordinary fermion (with annihilation operator \( f \)) can be split into two Majorana fermions, so \( N \) ordinary fermions are equivalent to \( 2N \) Majorana fermions
  \[ f_1 = \frac{(c_1+i c_2)}{2}, \quad f_2 = \frac{(c_3+i c_4)}{2}, \quad \ldots \]

Majorana fermions exist in spinless p-wave superconductors


An ordinary s-wave superconductor deposited on topological insulator hosts Majorana states at vortices

Ref: L. Fu and C. L. Kane, PRL 100, 096407 (2008)

A semiconductor with strong spin-orbit coupling (e.g., InAs) coupled to superconducting leads produce Majorana fermions at the ends of the wire

1-D p-wave superconducting chain

\[
H = \sum_{n=1}^{N} \left[ -w (f_{n}^{\dagger} f_{n+1} + f_{n+1}^{\dagger} f_{n}) + \Delta (f_{n} f_{n+1} + f_{n+1}^{\dagger} f_{n}^{\dagger}) \right] - \sum_{n=1}^{N} \mu (f_{n}^{\dagger} f_{n} - 1/2)
\]

\[a_{n} = f_{n} + f_{n}^{\dagger}, b_{n} = -i (f_{n} - f_{n}^{\dagger})\]

Majorana operator

\[
H = \frac{i}{2} \sum_{n=1}^{N-1} \left[ (-w + \Delta) a_{n} b_{n+1} + (w + \Delta) b_{n} a_{n+1} \right] - \frac{i}{2} \sum_{n=1}^{N} \mu a_{n} b_{n}
\]

end modes

Case A:

\[ w = \Delta \text{ and } \mu = 0 \]

Case B:

\[ w = -\Delta \text{ and } \mu = 0 \]

Case C:

\[ w = \Delta = 0 \text{ and } \mu \neq 0 \]
Phase diagram and end modes

- **Left panel:** I and II $\rightarrow$ topologically non-trivial phases, III $\rightarrow$ topologically trivial phase
- **Number of zero-energy Majorana modes** is the topological invariant of topological phases
- **Right panel:** Two isolated Majorana states are in two edges of a 100-sites open Majorana chain in phase I ($\mu = 0.0$, $\Delta = 0.1$ and $w = 1.0$). $a_n$ ($b_n$) is non-zero for the left (right) end of the chain respectively

Non-equilibrium dynamics of edge Majorana and adiabatic transport ...
Quenching of the Majorana chain

- **Quenching scheme:** $\Delta(t) = -1 + 2t/\tau$ with rate $\tau \gg 1$, $\mu = 0$ and $t \in [0, \tau]$, crosses QCP $\Delta = 0$ at $t = \tau/2$.

- **Initial state** $|\Psi(0)\rangle$ can be a bulk state or an edge Majorana for $\xi(0) = -1$ within the phase II.

- Probability of Majorana getting excited to the positive energy band $P_{\text{def}}$ (negative energy band $P_{\text{neg}}$) in phase I

  $$P_{\text{def(neg)}} = \sum_{\epsilon^+ > 0 (\epsilon^- < 0)} |\langle \epsilon^+(-) | \Psi(\tau) \rangle|^2$$

  where $|\Psi(\tau)\rangle$ is the time-evolved initial state.

- The Majorana excitation probability $P_m = |\langle \epsilon_0 | \Psi(\tau) \rangle|^2$ where $|\epsilon_0\rangle$ is the final equilibrium edge Majorana in the Phase I.

Violation of Kibble-Zurek scaling and no adiabatic transport

- Periodic chain or initial interior bulk state of an open chain $\Rightarrow$ expected KZ scaling: $n \sim \tau^{-\frac{\nu_d}{\nu z+1}} \sim \tau^{-1/2}$, $\nu = z = 1$

- Edge state $\Rightarrow$ deviates from the usual KZ scaling and is non-universal

- $P_{\text{def}} = P_{\text{neg}} = 0.5$ and $P_m = 0$

- Completely fuses in the positive and negative energy bands, adiabatic transport of Majorana ($b_1 \rightarrow b_N$ or $a_N \rightarrow a_1$) is impossible

Complete delocalization of edge Majorana throughout the chain

Equilibrium left edge MF at QCP:
\[ |L\rangle \approx (b_1 + b_3 + b_5 + \cdots, b_N)|\Omega\rangle \approx \frac{1}{\sqrt{2}}(|\frac{\pi}{2}\rangle_+ + |\frac{\pi}{2}\rangle_-) \]

Could the edge Majorana be transported...
Application of complex hopping term

\[ H = \sum_{n=1}^{N} \left[ -w_0 e^{i\phi} f_n^\dagger f_{n+1} - w_0 e^{-i\phi} f_{n+1}^\dagger f_n + \Delta (f_n f_{n+1} + f_{n+1}^\dagger f_n^\dagger) \right] - \sum_{n=1}^{N} \mu (f_n^\dagger f_n - 1/2) \]

- The Hamiltonian in the momentum space takes the form
  \[ h_k = (2w_0 \sin \phi \sin k) I - (2w_0 \cos \phi \cos k + \mu) \sigma^z + (2\Delta \sin k) \sigma^y \]

The Hamiltonian in terms of Majorana fermions:

\[ H = -i \frac{1}{2} \sum_{n=1}^{N-1} \left[ w_0 \cos \phi \left( a_n b_{n+1} + a_n b_{n-1} \right) - \Delta (a_n b_{n+1} - a_n b_{n-1}) \right] \]

\[ + w_0 \sin \phi \left( a_n a_{n+1} + b_n b_{n+1} \right) + \sum_{n=1}^{N} \mu a_n b_n \]

Under time-reversal \( a_n \to a_n \) and \( b_n \to -b_n \), operators like \( ia_m a_n \) or \( ib_m b_n \) break time-reversal symmetry

One can locate the QCPs given by the relation

\[(2w_0 \cos \phi \cos k_c + \mu)^2 + 4\Delta^2 \sin^2 k_c = 4w_0^2 \sin^2 \phi \sin^2 k_c\]

Two vertical lines (horizontal lines) given by \(\mu/w_0 = \pm 2 \cos \phi\)

\((\Delta/w_0 = \pm \sin \phi)\)
Quenching dynamics of an initial edge Majorana

- **Address:**
  Is adiabatic transport of edge Majorana from one topological phase to the other across the extended quantum critical region (from $\Delta = -\sin \phi$ to $\Delta = \sin \phi$) possible?

- **Observations:**
  1. The $P_m$ becomes finite for some values of $\tau$
  2. The adiabatic transport happens for set of values above a threshold value of transit time ($\Delta t_{th}$) through the gapless region
Probability of Majorana

Exists a characteristic $\tau$ (denoted by $\tau_c$) for which there is the first significant dip (peak) in $P_{\text{def}}$ ($P_m$) followed by a few drops (peaks)

$N = 100$
Optimum transit time

- Define **two instants of time** when the system enters and leaves the gapless phase as $t_e$ and $t_o$, respectively and the **transit time**
  \[ \Delta t = t_0 - t_e = \tau \sin \phi \]

- Exists an **optimum value of transit time** which is independent of $\phi$
  \[ \Delta t_{op} = \tau_c \sin \phi = \frac{\tau_c W_d}{2} \]

Clearly, $\tau_c$ depends on $\phi$ and decreases as it increases from 0 to $\pi/2$; $\tau_c \to \infty$; when $\phi \to 0$, satisfy Bermudez et al’s result
Optimum transit time and system size

- $\Delta t_{\text{op}} \propto N$
- Energy difference between two consecutive levels decreases with $N$, as a result relaxation time of the system increases and then $\tau_c$ increases linearly with system size $N$
- For few peaks following the first peak of $P_m$ exists $\Delta t_{\text{op}1}, \Delta t_{\text{op}2}, \cdots, \Delta t_{\text{op}n}$ ($\Delta t_{\text{op}(n-1)} < \Delta t_{\text{op}n}$)
- $\tau_c$ depends on the quenching length, but $\Delta t_{\text{op}}$ remains fixed..it is a universal constant which depends only on $N$
Does Majorana state persist after the quenching is stopped?

- For $t > \tau$, $|\Psi(\tau)\rangle$ evolves as $\exp(-iH(\Delta = 1)t)|\Psi(\tau)\rangle$

- $P_m(t > \tau) = |\langle \epsilon_0 | \exp(-iH(\Delta = 1)t)|\Psi(\tau)\rangle|^2$ does not change with time $t$

- A finite probability of the edge Majorana in phase I exists even for $t > \tau$
Energy levels: **Band Inversion**

- Band inversion occurs inside the zone bounded by $\Delta/w_0 = \pm \sin \phi$ (gapless region under PBC)
- Number of inverted energy levels increases with $N$ and $\phi$
- Is it going give any new feature in the dynamics of edge Majorana?
Plausible conjecture

- Exists a **threshold value** of $\Delta t_{th} = \Delta t_{op1}$, above which Majorana starts delocalizing maximally within the inverted band

- This phenomena leads to **adiabatic tunneling** of edge Majorana

- For $\Delta t < \Delta t_{th}$ edge Majorana diffuses into the interior bulk, no possibility of recombination

- First significant peak height of $P_m$ decreases with increasing $N$ and $\phi$, interaction with more number of inverted energy levels

Young Quantum-2015, HRI
A phase factor in the hopping term results an extended gapless region and breaks time-reversal symmetry of the Hamiltonian that leads to give zero-energy Majoranas involving both $a$ and $b$.

The extended gapless region is insensitive to quenching of a periodic chain, but show interesting behavior for edge state quenching.

Exists inverted energy levels near zero-energy for non-zero $\phi$ and for $\Delta t \geq \Delta t_{th}$, edge Majorana can be transported adiabatically from one topological phase to the other.

Adibatic transport of edge Majorana becomes more probable with a less number of inverted bands within the gapless region.
Thank You
Adiabatic dynamics and Kibble-Zurek scaling

A driven quantum many body system has two time scales

(I) Relaxation time: Inverse of minimum gap
(II) Time scale for driving the Hamiltonian

Adiabatic dynamics must follow the relation:
Relaxation time $<<$ Time scale of driving

Relaxation time diverges close to the critical point: Dynamics becomes diabatic.

Density of defect scales as

$$n \sim \tau^{-\nu d \over \nu z + 1}$$

(Kibble-Zurek (KZ) scaling)

where, $d \to$ dimension of the system, $\nu \to$ correlation length exponent and $z \to$ dynamical exponent

The Hamiltonian $h_k$ with $w_0 = 1$ following an appropriate unitary transformation takes the form

$$h_k(t) = (2 \sin \phi \sin k) I + (2\Delta(t) \sin k) \sigma^z + (2 \cos \phi \cos k) \sigma^y$$

$\Delta(t) = -1 + 2t/\tau$ with quench time $\tau \gg 1$ and $t \in [0, \tau]$ along the path $\mu = 0$

General state for a given $k-$ mode $|\psi_k(t)\rangle = u_k(t)|1_k\rangle + v_k(t)|2_k\rangle$

Identity part of the Hamiltonian is irrelevant for dynamics; time evolution is dictated by LZ term

Probability of excitation for each mode $p_k = e^{-2\pi \gamma_k}$

where, $\gamma_k = \delta_k^2 / \left| \frac{d}{dt} (E_1 - E_2) \right|$, $\delta_k = 2 \cos \phi \cos k$ and $E_{1,2} = \pm 2\Delta(t) \sin k$

Defect density in the final state is given by
\[
n = \frac{2\pi}{N} \int_{-\pi}^{\pi} p_k dk \sim \frac{1}{\pi} \frac{1}{\cos \phi \sqrt{\tau}}
\]

- The maximum contribution in the above integration comes from \( k_c = \pi/2 \) where the energy gap of LZ part vanishes for \( \Delta = 0 \)
- Dynamics is completely insensitive to gapless region
- KZ scaling is satisfied analytically as well as numerically, note that \( \tau \) gets renormalized to \( \tau_{\text{eff}} = \tau \cos^2 \phi \)