Characterizing quantum correlations in the framework of generalized nonsignaling theory (GNST)

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February 23, 2015

In GNST, the states are described by JPD: $P(ab|AB)$.

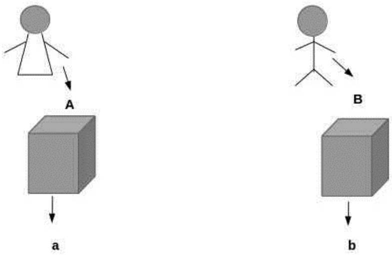


Figure : Two distant parties have access to a black box: Alice and Bob input $A, B \in \{0, 1, 2, ...\}$ and get outputs $a, b \in \{0, 1, 2, 3, ...\}$

Nonlocal and nonsignaling correlations

NS contraints: $P(a|AB) = \sum_b P(ab|AB) = P(a|A)$ and $P(b|AB) = \sum_a P(ab|AB) = P(b|B)$.
Nonlocality: $P(ab|AB) \neq \sum_\lambda p_\lambda P_\lambda(a|A) P_\lambda(b|B)$.

Bell-CHSH Scenario:

NS boxes with 2 binary inputs and outputs. $P(a_m, b_n|A_i, B_j) =$

$$\begin{pmatrix} P(a_0,b_0|A_0,B_0) & P(a_0,b_1|A_0,B_0) & P(a_1,b_0|A_0,B_0) & P(a_1,b_1|A_0,B_0) \\ P(a_0,b_0|A_0,B_1) & P(a_0,b_1|A_0,B_1) & P(a_1,b_0|A_0,B_1) & P(a_1,b_1|A_0,B_1) \\ P(a_0,b_0|A_1,B_0) & P(a_0,b_1|A_1,B_0) & P(a_1,b_0|A_1,B_0) & P(a_1,b_1|A_1,B_0) \\ P(a_0,b_0|A_1,B_1) & P(a_0,b_1|A_1,B_1) & P(a_1,b_0|A_1,B_1) & P(a_1,b_1|A_1,B_1) \end{pmatrix}$$

The PR-box (Popescu and Rohrlich, Found. Phys 1994),

$$P_{PR} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

maximally violates the Bell-CHSH inequality (CHSH, PRL 1969):

$$\mathcal{B} := \langle A_0B_0\rangle + \langle A_0B_1\rangle + \langle A_1B_0\rangle - \langle A_1B_1\rangle \leq 2.$$

Logic contradiction of PR-box with local realism:

$$\begin{aligned} A_0B_0 = A_0B_1 = A_1B_0 = 1 \\ A_1B_1 = -1 \end{aligned} \quad (1)$$

The NS boxes forms an 8-dim convex polytope.

$$P_{PR}^{\alpha\beta\gamma}(a_m, b_n|A_i, B_j) = \begin{cases} \frac{1}{2}, & m \oplus n = ij \oplus \alpha i \oplus \beta j \oplus \gamma \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

$$P_D^{\alpha\beta\gamma\epsilon} = \begin{cases} 1, & m = \alpha i \oplus \beta \\ & n = \gamma j \oplus \epsilon \\ 0, & \text{otherwise} \end{cases} \quad (3)$$

Here $\alpha, \beta, \gamma, \epsilon \in \{0, 1\}$ and $\oplus$ denotes addition modulo 2.

LRO

Alice changing her input $i \to i \oplus 1$, and changing her output conditioned on the input: $m \to m \oplus \alpha i \oplus \beta$.

Quantum scenario: $A_i = \vec{a}_i \cdot \vec{\sigma}$ and $B_j = \vec{b}_j \cdot \vec{\sigma}$ on $\rho_{AB} \in \mathcal{B}(H_2 \otimes H_2)$

$P(a_m, b_n|A_i, B_j) = \mathrm{tr}\left(\rho_{AB} \Pi_{a_m}^{A_i} \otimes \Pi_{b_n}^{B_j}\right)$, where $\Pi_{A_i}^{a_m}$ and $\Pi_{B_j}^{b_n}$, are the projectors generating binary outcomes $a_m, b_n \in \{-1, 1\}$.

$$P = \sum_{k=0}^{7} p_k P_{PR}^{k} + \sum_{l=0}^{15} q_l P_D^l ; \sum_k p_k + \sum_l q_l = 1. \tag{4}$$

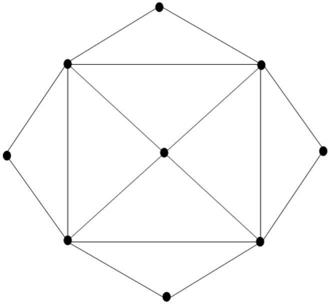


Figure : NS polytope: Barrett et al PRA 2005

Bell polytope ($\mathcal{L}$)

$\mathcal{L}$ is a convex hull of the 16 deterministic boxes.

If $P(a_m, b_n|A_i, B_j) \in \mathcal{L}$,

$$P(a_m, b_n|A_i, B_j) = \sum_{l=0}^{15} q_l P_D^l; \sum_l q_l = 1, \Longleftrightarrow \mathcal{B}_{\alpha\beta\gamma} \leq 2^{a} \quad (5)$$

[a] Fine PRL 1982

Here, $P_D^l = P_D^l(a_m|A_i)P_D^l(b_n|B_j)$.

$$\begin{aligned} \mathcal{B}_{\alpha\beta\gamma} &:= (-1)^{\gamma} \langle A_0 B_0 \rangle + (-1)^{\beta\oplus\gamma} \langle A_0 B_1 \rangle \\ &+ (-1)^{\alpha\oplus\gamma} \langle A_1 B_0 \rangle + (-1)^{\alpha\oplus\beta\oplus\gamma\oplus 1} \langle A_1 B_1 \rangle \end{aligned}$$

where $\langle A_i B_j \rangle = \sum_{mn} (-1)^{m\oplus n} P(a_m, b_n|A_i, B_j)$.

My goal:

- Local correlations arising from the entangled states and nonzero quantum discord states have nonclassicality.
  I introduced the measures Bell discord (BD) and Mermin discord (MD) to quantify nonlocality and contextuality of any quantum correlation.
- I found 3-decomposition fact for NS boxes with two inputs and two outputs using geometry of nonsignaling polytope w.r.t BD and MD.

$$P = G' P_{PR}^{\mathcal{G}=4} + Q' P_{ML}^{\mathcal{Q}=2} + (1 - G' - Q') P_{\mathcal{G}=0}^{\mathcal{Q}=0} \tag{6}$$

Nonlocality and Bell discord

Nonlocality of quantum theory implies that the correlations arises from incompatible measurements and entanglement. The Bell state,

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad (7)$$

gives rise to the Tsirelson bound [1], $\mathcal{B} = 2\sqrt{2}$, for the measurement observables:

$$A_0 = \sigma_x, A_1 = \sigma_y, B_0 = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_y), B_1 = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y). \quad (8)$$

The JPD that achieves the Tsirelson bound can be decomposed into PR-box and white noise:

$$P = p_{PR}P_{PR} + (1 - p_{PR})P_N, \quad p_{PR} = \frac{1}{\sqrt{2}} \quad (9)$$

[1] Tsirelson, Lett. Math. Phys. 1980

Definition

A correlation is said to have Bell discord iff it admits a decomposition that has irreducible PR-box component.

For the measurements that gives rise to the Tsirelson bound, the pure entangled states,

$$|\psi\rangle = \cos\theta\,|00\rangle + \sin\theta\,|11\rangle \quad 0 \leq \theta \leq \frac{\pi}{4} \tag{10}$$

and the Werner states,

$$\rho_W = p\,|\psi^+\rangle\langle\psi^+| + (1-p)\mathrm{I} \tag{11}$$

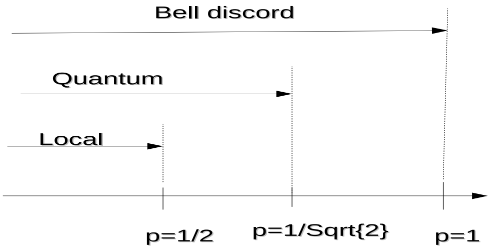
(entangled if $p > \frac{1}{3}$ and has Quantum discord if $p > 0$ (Ollivier and Zurek, PRL, 2001)), admit the decomposition

$$P = p_{PR}P_{PR} + (1-p_{PR})P_N, \quad p_{PR} = \frac{\sin 2\theta}{\sqrt{2}}, \frac{p}{\sqrt{2}} \tag{12}$$

These correlations are local if $p_{PR} \leq \frac{1}{2}$ and have Bell discord if $p_{PR} > 0$.

Nonclassicality due to Bell discord=incompatible measurements+quantum discord→PR-box component.

Fig: The isotropic PR-box: $P = pP_{PR} + (1-p)P_N$.



Bell discord, $0 \leq \mathcal{G} \leq 4$

$$\mathcal{G} = \min_{i=1}^{3} \mathcal{G}_i \tag{13}$$

where $\mathcal{G}_1 := \Big||\mathcal{B}_{00} - \mathcal{B}_{01}| - |\mathcal{B}_{10} - \mathcal{B}_{11}|\Big|$. $\mathcal{G} = 0$ for the deterministic boxes and $\mathcal{G} = 4$ for the PR-boxes.

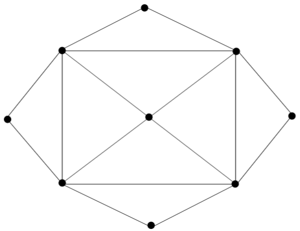


Figure : $\mathcal{G}$ divides the local polytope into a $\mathcal{G} > 0$ region and $\mathcal{G} = 0$ nonconvex region.

$$P = \sum_{k=0}^{7} p_k P_{PR}^{k} + \sum_{l=0}^{15} q_l P_D^l; \sum_k p_k + \sum_l q_l = 1. \quad (14)$$

Unequal mixture of any two PR-box can be reduced to convex mixture of a single PR-box and a Bell-local box:
$pP_{PR}^{000} + qP_{PR}^{001} = (p-q)P_{PR}^{000} + 2q\frac{P_{PR}^{000}+P_{PR}^{001}}{2}$

Theorem

Any NS box (nonlocal, or not) can be written as,

$$P = \mathcal{G}' P_{PR}^{\alpha\beta\gamma} + (1-\mathcal{G}')P_L^{\mathcal{G}=0}. \quad (15)$$

Local contextuality and Mermin discord

Nonlocality is one of the manifestations of contextuality.
The Bell state gives rise to local-contextual correlation (Mermin box),

$$P_M = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

for the measurement observables,

$$A_0 = \sigma_x, A_1 = \sigma_y, B_0 = \sigma_x, B_1 = \sigma_y \tag{16}$$

KS paradox (Peres, PLA 1990):

$$\begin{aligned} \sigma_x \otimes \sigma_x |\psi^+\rangle &= |\psi^+\rangle \\ \sigma_y \otimes \sigma_y |\psi^+\rangle &= -|\psi^+\rangle \\ (\sigma_x \otimes \sigma_y)(\sigma_y \otimes \sigma_x) |\psi^+\rangle &= |\psi^+\rangle . \end{aligned} \tag{17}$$

$$A_0 B_0 = 1 \quad A_1 B_1 = -1$$
$$A_0 B_1 A_1 B_0 = 1 \tag{18}$$

For the measurements that gives rise to KS paradox, the pure entangled states and the Werner states admit the decomposition,

$$P = p_M P_M + (1 - p_M) P_N, \tag{19}$$

with $p_M = \sin 2\theta$ and $p_M = p$.

Definition

A correlation is said to have Mermin discord iff it admits a decomposition that has irreducible Mermin box component.

Mermin discord, $0 \leq \mathcal{Q} \leq 2$

$$\mathcal{Q} = \min \mathcal{Q}_i \tag{20}$$

where $\mathcal{Q}_1 := \Big| |\mathcal{M}_{00} - \mathcal{M}_{01}| - |\mathcal{M}_{10} - \mathcal{M}_{11}| \Big|$. $\mathcal{Q} = 0$ for any deterministic box and PR box. $\mathcal{Q} = 2$ for the Mermin boxes.

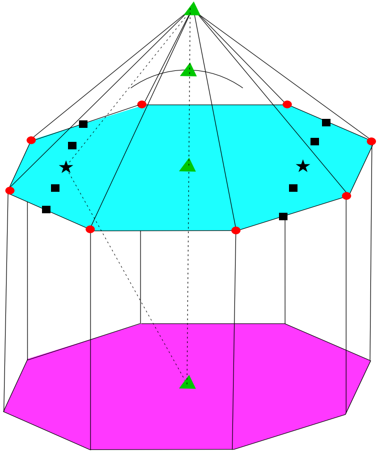
Here,

$$\begin{aligned} \mathcal{M}_{\alpha\beta} &:= (\alpha \oplus \beta \oplus 1) |(-1)^{\alpha} \langle A_0 B_1 \rangle + (-1)^{\alpha \oplus \beta} \langle A_1 B_0 \rangle | \\ &+ (\alpha \oplus \beta) |(-1)^{\alpha} \langle A_0 B_0 \rangle + (-1)^{\alpha \oplus \beta} \langle A_1 B_1 \rangle |, \end{aligned} \tag{21}$$

For the measurements: $A_0 = \sigma_x$, $A_1 = \sigma_y$, $B_0 = \sqrt{p}\sigma_x - \sqrt{1-p}\sigma_y$ & $B_1 = \sqrt{1-p}\sigma_x + \sqrt{p}\sigma_y$, where $\frac{1}{2} \leq p \leq 1$, the Bell state, $|\psi^+\rangle$, gives rise to the following correlation,

$$P = \mathcal{G}' P_{PR} + \mathcal{Q}' P_M + (1 - \mathcal{G}' - \mathcal{Q}') P_N, \tag{22}$$

where $\mathcal{G}' = \sqrt{1-p}$ and $\mathcal{Q}' = \sqrt{p} - \sqrt{1-p}$.



3-decomposition fact

$$P = \mathcal{G}' P_{PR}^{\alpha\beta\gamma} + (1 - \mathcal{G}') P_L^{\mathcal{G}=0}. \quad (23)$$

$\mathcal{Q}$ divides the $\mathcal{G} = 0$ polytope into $\mathcal{Q} > 0$ region and $\mathcal{G} = \mathcal{Q} = 0$ polytope.

Theorem

Any NS box (nonlocal, or not) can be written as,

$$P = \mathcal{G}' P_{PR}^{\alpha\beta\gamma} + \mathcal{Q}' P_{\mathcal{Q}=2} + (1 - \mathcal{G}' - \mathcal{Q}') P_{\mathcal{Q}=0}^{\mathcal{G}=0}. \quad (24)$$

$\mathcal{G}(P) = 4\mathcal{G}'$ *and* $\mathcal{Q}(P) = 2\mathcal{Q}'$ *which implies that* $\mathcal{G}'$ *and* $\mathcal{Q}'$ *are symmetric and invariant under LRO.*

Monogamy:

$$\mathcal{G} + 2\mathcal{Q} \leq 4. \quad (25)$$

Bell and Mermin discord are semi-device-independent witnesses for nonclassicality of nonzero quantum discord states

$$\rho_{CQ} = \sum_{i=0}^{1} p_i |i\rangle\langle i| \otimes \chi_i, \tag{26}$$

$$\rho_{QC} = \sum_{j=0}^{1} p_j \phi_j \otimes |j\rangle\langle j|, \tag{27}$$

where, $\{|i\rangle\}$ and $\{|j\rangle\}$ are the orthonormal sets on Alice's and Bob's side and χ_i and ϕ_i are the quantum states.

$$\rho_{CQ} = \frac{p_0}{4}\left(\mathbf{1} + \hat{\mathbf{r}} \cdot \tilde{\sigma}\right) \otimes \left(\mathbf{1} + \tilde{\mathbf{s}}_\mathbf{0} \cdot \tilde{\sigma}\right) + \frac{p_1}{4}\left(\mathbf{1} - \hat{\mathbf{r}} \cdot \tilde{\sigma}\right) \otimes \left(\mathbf{1} + \tilde{\mathbf{s}}_\mathbf{1} \cdot \tilde{\sigma}\right) \tag{28}$$

$$\langle A_i B_j \rangle = (\hat{a}_i \cdot \hat{r}) \left(\hat{b_j} \cdot (p_0 \vec{s}_0 - p_1 \vec{s}_1)\right) \tag{29}$$

For the Werner states, $\langle AB \rangle = p(\hat{a} \cdot \hat{b})$.

Tripartite Scenario

The set of tripartite NS boxes with 2 binary inputs and outputs forms a convex polytope in 26-dimensional space with 53728 extremals. I restricted to the Svetlichny-box polytope, $P(a_m, b_n, c_o|A_i, B_j, C_k)$

$$= \sum_{i=0}^{15} p_i P^i_{Sv} + \sum_{i=0}^{15} q_i P^i_{12} + \sum_{i=0}^{15} r_i P^i_{13} + \sum_{i=0}^{15} s_i P^i_{23} + \sum_{j=0}^{63} t_j P^j_D, \quad (30)$$

A correlation is said to be Bell nonlocal if

$$P(a_m, b_n, c_o|A_i, B_j, C_k) \neq \sum_\lambda p_\lambda P_\lambda(a_m|A_i) P_\lambda(b_n|B_j) P_\lambda(c_k|C_k). \quad (31)$$

A correlation is said to be genuinely nonlocal if

$$\begin{aligned} P(a_m, b_n, c_o|A_i, B_j, C_k) &\neq p_1 \sum_\lambda p_\lambda P_\lambda^{AB|C} + p_2 \sum_\lambda q_\lambda P_\lambda^{AC|B} \\ &+ p_3 \sum_\lambda r_\lambda P_\lambda^{A|BC}. \end{aligned} \quad (32)$$

Svetlichny discord

The GHZ state, $|\psi_{GHZ}\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, violates the SI (Svetlichny 1987),

$$\begin{aligned} \mathcal{S} \;&:=\; \langle A_0B_0C_0\rangle+\langle A_0B_0C_1\rangle+\langle A_0B_1C_0\rangle-\langle A_0B_1C_1\rangle+ \\ &\langle A_1B_0C_0\rangle-\langle A_1B_0C_1\rangle-\langle A_1B_1C_0\rangle-\langle A_1B_1C_1\rangle \le 4, \end{aligned} \quad (33)$$

to the quantum bound, $\mathcal{S} = 4\sqrt{2}$, for the measurements,

$$A_0 = \sigma_x, A_1 = \sigma_y, B_0 = \sigma_x, B_1 = \sigma_y, C_k = \frac{1}{\sqrt{2}}\left(\sigma_x - (-1)^k\sigma_y\right) \quad (34)$$

The JPD that gives this bound can be decomposed as the convex mixture of the Svetlichny box and white noise:

$$P = p_{Sv}P^{0000}_{Sv} + (1 - p_{Sv})P_N, \quad (35)$$

with $p_{Sv} = \frac{1}{\sqrt{2}}$; the GGHZ states, $\cos\theta\,|000\rangle + \sin\theta\,|111\rangle$, have $p_{Sv} = \sin 2\theta$. Notice that the correlation is local if $p_{Sv} \le \frac{1}{2}$.

For the measurements,

$$A_0 = \sigma_x, A_1 = \sigma_y, B_0 = \sigma_x, B_1 = \sigma_y, C_0 = \sigma_x, C_1 = \sigma_y, \tag{36}$$

the GHZ state maximally violates the MI (Mermin, 1990),

$$\mathcal{M} := |\langle A_0 B_0 C_0 \rangle - \langle A_0 B_1 C_1 \rangle - \langle A_1 B_0 C_1 \rangle - \langle A_1 B_1 C_0 \rangle| \leq 2, \tag{37}$$

i.e., it gives $\mathcal{M} = 4$. The correlations can be decomposed as follows,

$$P_M = \frac{1}{4} \sum_{\lambda=1}^{4} P_\lambda(a_m|A_i) P_\lambda(b_n, c_o|B_j, C_k), \tag{38}$$

GHZ paradox:

$$\begin{aligned} \sigma_y \otimes \sigma_y \otimes \sigma_x |\psi_{GHZ}\rangle &= -|\psi_{GHZ}\rangle \\ \sigma_x \otimes \sigma_y \otimes \sigma_y |\psi_{GHZ}\rangle &= -|\psi_{GHZ}\rangle \\ \sigma_y \otimes \sigma_x \otimes \sigma_y |\psi_{GHZ}\rangle &= -|\psi_{GHZ}\rangle \\ \sigma_x \otimes \sigma_x \otimes \sigma_x |\psi_{GHZ}\rangle &= |\psi_{GHZ}\rangle \end{aligned} \tag{39}$$

Mermin box:

$$P_M = \frac{1}{2} \left(P_{Sv}^{0000} + P_{Sv}^{1111} \right). \tag{40}$$

- Svetlichny discord: $0 \leq \mathcal{G} \leq 8$; $\mathcal{G} = 0$ for all the bipartite PR-boxes and local deterministic boxes, whereas any Svetlichny box has $\mathcal{G} = 8$.
- Mermin discord: $0 \leq \mathcal{Q} \leq 4$; $\mathcal{Q} = 0$ for all the local deterministic boxes, bipartite PR-boxes and Svetlichny boxes, whereas $\mathcal{Q} = 4$ for the Mermin boxes.

3-decomposition fact of quantum boxes

Any box (genuinely nonlocal, or not) that belongs to the Svetlichny box polytope can be written as,

$$P = \mathcal{G}' P_{Sv}^{\alpha\beta\gamma\epsilon} + \mathcal{Q}' P_{M}^{\alpha\beta\gamma\epsilon} + (1 - \mathcal{G}' - \mathcal{Q}') P_{\mathcal{G}=0}^{\mathcal{Q}=0} \quad (41)$$

Genuinely nonclassical states which cannot be decomposed in the classical-quantum form,

$$\rho_{CQ} = \sum_i \rho_i^A \otimes \rho_i^{BC}, \quad (42)$$

and the permutations, can give rise to nonzero $\mathcal{G}'$ and $\mathcal{Q}'$.

$$\rho_W = p \left|\psi_{GHZ}\right\rangle \left\langle\psi_{GHZ}\right| + (1-p)\mathbf{1} \tag{43}$$

- The Werner states are separable iff $p \leq 0.2$, biseparable iff $0.2 < p \leq 0.429$ and genuinely entangled iff $p > 0.429$
- $P = \frac{p}{\sqrt{2}}P_{Sv} + \left(1 - \frac{p}{\sqrt{2}}\right)P_N$; local if $p \leq \frac{1}{\sqrt{2}}$

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Thank you!