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Adiabatic Berry Phase and Hannay Angle for Open Paths

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We obtain the adiabatic Berry phase by defining a generalised gauge potential whose line integral gives the phase holonomy for arbitrary evolutions of parameters. Keeping in mind that for classical integrable systems it is hardly clear how to obtain the open-path Hannay angle, we establish a connection between the open-path Berry phase and Hannay angle by using the parametrised coherent state approach. Using the semiclassical wavefunction we analyse the open-path Berry phase and obtain the open-path Hannay angle. Further, by expressing the adiabatic Berry phase in terms of the commutator of instantaneous projectors with its differential and using Wigner representation of operators we obtain the Poisson bracket between the distribution function and its differential. This enables us to talk about the classical limit of the phase holonomy which yields the angle holonomy for open paths. An operational definition of the Hannay angle is provided based on the idea of the classical limit of the quantum mechanical inner product. A probable application of the open-path Berry phase and Hannay angle to the wave-packet revival phenomena is also pointed out. © 1998 Academic Press

1. INTRODUCTION

In recent years, the quantal phase holonomy [1] of purely geometrical origin has played an important and fundamental role in diverse areas of physics. Berry [2] discovered this in the quantum adiabatic context, where the quantal eigenstate acquires an extra phase when the Hamiltonian of the system is adiabatically transported arround a closed path in parameter space. At the classical level there is a similar effect, namely, the angle holonomy, discovered by Hannay [3]. For integrable systems (where it is possible to write the Hamiltonian of the system in terms of action and angle variables), the Hannay angle is nothing but an extra angle shift picked up by the angle variables of the classical system when the parameters undergo adiabatic change along a closed path in the parameter space. After the importance of Berry's discovery was realised in many areas of physics, it was liberated from its restrictions to adiabatic, periodic variations of Hamiltonian evolutions. Aharonov and Anandan [4] showed the existence of the geometric phases for non-adiabatic, cyclic evolutions of quantal wavefunctions. Samuel and Bhandari [5] generalised the idea of phase holonomy for non-cyclic, non-unitary evolutions of quantum systems. Mukunda and Simon [6] have generalised the

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concept of geometric phase using kinematic concepts of the ray space. Recently, the present author generalised it further to the case of non-cyclic, non-unitary, and non-Schrödinger evolutions of the quantum systems [15]. Notwithstanding the wide generalisation of the Berry phase, its classical counterpart the Hannay angle has not been generalised further except for non-adiabatic cases. Berry and Hannay [8] have obtained the classical non-adiabatic angle as the holonomy of a non-trivial connection in the phase-space bundle. The Hannay angle can also be understood as an angle shift in transporting a classical tori in phase space [9]. Therefore, any attempt to generalise and understand the classical angle holonomy for open paths is quite challenging.

In this paper we generalise the Berry phase and Hannay angle for an adiabatically evolving system with non-cyclic variation of the external paramaters of the Hamiltonian. Before achieving that we provide a gauge potential description of the open-path Berry phase. This defines a quantum one-form whose line integral gives the Berry phase during an arbitrary variations of external parameters. Using the parametrised coherent state approach we establish a connection between the Berry phase and open-path Hannay angle. Also, we obtain the open-path Berry phase in the semiclassical limit and relate it to the Hannay angle. Further, we express the quantum one-form in terms of instantaneous projection operators and study its classical limit using the correspondence between the quantum commutator and Poisson bracket. Here, we have used the Wigner representation of quantum mechanical distribution function and phase space functions. The generalisation of the Hanny angle will have many important applications such as wave-packet revivals [10], field theoretical models with fermions, and Grasmannian systems [11]. The present work will be a first step in this direction. We will not give a treatment of the open-path Hannay angle based on the classical Hamiltonian and its cannonical transformation to action-angle variables, rather we will define the adiabatic Berry phase for open paths in parameter space and obtain the Hannay angle as a semiclassical limit of the former. For arriving at the Hannay angle the following result will be invoked: The connection between the Hannay angle and Berry phase [12] is valid not only for the adiabatic closed-excursions but also for the open-excursions in the parameter space. The reason for doing this is that there is a difficulty in attacking the problem purely at a classical level. For integrable, bounded motions of classical systems action variables are the classical, adiabatic invariants (in the quantum case, the quantum number is an adiabatic invariant). These angle variables have some unavoidable arbitrariness in their definition and they cannot be compared belonging to distinct initial and final Hamiltonians [3]. They can be compared, however, if the Hamiltonian is varied arround a closed loop in parameter space so that the initial and final Hamiltonians are the same. Then one can make the Hannay angle coordinate independent (in the quantum case this is equivalent to making the Berry phase gauge invariant). If we wish to define the Hannay angle for open paths from classical considerations we would face the problem. of comparing the angle variables belonging to distinct initial and final Hamiltonians. However, at a quantum mechanical level there is no problem in comparing the phases of two distinct (they do not form the same equivalence classes) initial and final non-orthogonal vectors. Therefore, it seems natural to define the quantal adiabatic Berry phase for open paths in parameter space and then analyse it within the semiclassical and classical limits. Towards this end, an application of the present work will be pointed out, where one can show the effect of the open-path Hannay angle on wave-packet revivals. The effect of Hannay angle on revivals has been recently discussed by Jarzynski [10] for cyclic variations of external parameters. In a sense, the application of the present formulation will be an extension of his prediction which says that the effect of adiabatic variation of parameters is to cause a displacement of the location at which the revived wave-packet appears, *even though the parameters do not return to their original value over the revival time.*

2. ADIABATIC BERRY PHASE FOR OPEN PATHS

2a. *Berry Phase for Closed Paths*

Before providing the generalised Berry phase formula, it is useful to recapitulate the standard Berry phase formula. Consider a quantum system which is bounded, integrable, and driven by a slowly changing Hamiltonian $H(\mathbf{R}(t))$; $\{\mathbf{R} = R_i\}$ is the set of externally controllable parameters. Then using the adiabatic approximations, the solution to the Schrödinger equation is given by

$$|\Psi(t)\rangle = \exp\left[-\frac{i}{\hbar}\int_0^t E_n(t)\,dt \exp(i\gamma_n(t))\right] |\Psi_n(\mathbf{R}(t))\rangle, \tag{1}$$

where the $|\Psi_n(\mathbf{R}(t))\rangle$'s are instantaneous eigenstates of the Hamiltonian with non-degenerate eigenvalues $E_n(t)$. This foregoing Eq. (1) says that the system remains in the eigenstate with quantum number n apart from phase factors. The first phase factor is the usual dynamical one. The extra phase factor $\exp(i\gamma_n(t))$ becomes physically important and non-trivial only when the parameters are changed along a closed path over some time (large enough) T, such that $\mathbf{R}(T) = \mathbf{R}(0)$. Otherwise, these extra phases can always be chosen identically to be zero by choosing a different eigenfunction. The non-trivial phase is the Berry phase for closed paths in the parameter space, given by

$$\gamma_n(C) = i \oint_C \left\langle \Psi_n(\mathbf{R}) \,|\, \nabla\Psi_n(\mathbf{R}) \right\rangle \cdot d\mathbf{R} = \oint_C \mathbf{A}_n(\mathbf{R}) \cdot d\mathbf{R}. \tag{2}$$

This is nothing but the line integral of a vector potential $\mathbf{A}_n(\mathbf{R})$ (called the Berry potential or Berry one-form) arround the closed curve in parameter space and which can also be written as a surface integral of a vector field (two-form) where the surface is bounded by the closed curve C. As is well known, this is non-integrable in nature and depends only on the geometry of the path in the parameter space. In addition to this, it is gauge invariant. The phase $\gamma_n(C)$ is independent of the rate at which the circuit C is traversed, provided the adiabatic approximation holds. Therefore, the Berry phase is an essential ingredient of the adiabatic cyclic evolution of a quantum system.

2b. *Generalisation of the Berry Phase to Open Paths*

Suppose that the parameters which have been adiabatically changed along an arbitrary curve Γ do not come back to their original value after some time t_f. Can we still assign a geometric phase to such an adiabatically evolving quantum state? The answer is yes, though the phase, in this case, is not given by the expression

$$\gamma_n(\Gamma) = \int_{\mathbf{R}(0)}^{\mathbf{R}(t_f)} \mathbf{A}_n(\mathbf{R}) \cdot d\mathbf{R}. \tag{3}$$

In the past it has been mentioned incorrectly that the non-cyclic Berry phase would be still given by the above expression [13]. The reason being that the above expression is not gauge invariant under local gauge transformations of the eigenstates. We call an expression of the type (3) the "Berry term" which reduces to the Berry phase for a closed loop in parameter space. To obtain the Berry phase formula for open paths we have to take care of the contributions from the end points of the open path. When we do that the whole expression can be made gauge invariant.

The mathematical and physical basis underlying the open-path Berry phase formula can be given in terms of the fiber bundle descriptions of the adiabatically evolving eigenstates. As illustrated by Simon [1], the fiber bundle has a base space M (which is the space of parameters), has fibers (the set of phase factors, namely the group $U(1)$) and has the bundle space E (in which the adiabatic eigenstates exist). The bundle space E over M is defined by associating $\mathbf{R} \to |\Psi_n(\mathbf{R})\rangle$ given by $H(\mathbf{R})\,|\Psi_n(\mathbf{R})\rangle = E_n(\mathbf{R})\,|\Psi_n(\mathbf{R})\rangle$ with fibers $U(1)$. Geometrically we can imagine that the time evolution of the eigenstate is represented by a path in the bundle space E. The path in the bundle space can be constructed by the knowledge of path that is actually followed by the parameters in the base space. For example, in the case of cyclic change of parameters the path in the base space is a closed curve, whereas the path in the bundle space is an open one with the initial and final points belonging to the same fiber. However, if the parameters do not come back to their original value after some time t_f, then the base space path is an open path and correspondingly the lift of this is also an open path in the bundle space. But in this latter case the initial and final points of the bundle path are not on the same fiber. We are concerned here precisely with this type of adiabatic evolutions.

In general (irrespective of adiabaticity, cyclicity, and unitarity), when the initial and final states of the evolving quantal system belong to two different fibers, we can compare the phases by taking the inner product between them. This is in the spirit the Pancharatnam [14] way of defining the phase difference between two different polarisation states of light. However, the only restriction here is that the initial and final states should not be orthogonal to each other. Let $|\Psi(t)\rangle \in \mathscr{H}$ be the state of a system at some instant of time. During a non-cyclic evolution of the state vector in $\mathscr{H}$ it traces an open curve whose projection is also an open curve Γ: $\rho(0) \to \rho(t) \to \rho(t_f) \neq \rho(0)$ in the projective Hilbert space, where $\rho(t) = |\Psi(t)\rangle\langle\Psi(t)|$ is a pure state density operator. The total phase difference between the initial and final states is given by

$$\Phi_T = \arg\langle\Psi(t_i) \mid \Psi(t_f)\rangle = \arg\langle\Psi(0) \mid \Psi(t_f)\rangle. \tag{4}$$

Using the projective geometric structure of the Hilbert space, it has been shown by the present author [15] that the geometric phase during an arbitrary evolution of the quantum system is given by

$$\Phi_g = i \int \langle\chi(t) \mid d\chi(t)\rangle, \tag{5}$$

where $|\chi(t)\rangle$ is a "reference-section" defined from the actual state as $|\chi(t)\rangle = (\langle\Psi(t) \mid \Psi(0)\rangle)/(|\langle\Psi(t) \mid \Psi(0)\rangle|)\,|\Psi(t)\rangle$ and $i\langle\chi(t) \mid d\chi(t)\rangle$ is a connection-form defined over the projective Hilbert space of the quantum system. Thus, Φ_g can be regarded as the holonomy of the $U(1)$ bundle over the projective Hilbert space $\mathscr{P}$ of the quantum system.

When the quantum evolution is necessarily adiabatic and the open path arises from the adibatic evolution of external parameters, then we obtain the open-path Berry phase, which is given by

$$\gamma_n(\Gamma) = i \int_\Gamma \langle\chi_n(\mathbf{R}) \mid \nabla\chi_n(\mathbf{R})\rangle \cdot d\mathbf{R} = \int_\Gamma \Omega_n(\mathbf{R}) \cdot d\mathbf{R}, \tag{6}$$

where $|\chi_n(R)\rangle$ is the "reference-eigenstate" defined from the adiabatic eigenstate as $|\chi_n(\mathbf{R})\rangle = (\langle\Psi_n(\mathbf{R}) \mid \Psi_n(\mathbf{R}(0))\rangle)/|(\langle\Psi_n(\mathbf{R}) \mid \Psi_n(\mathbf{R}(0))\rangle|)\,|\Psi_n(\mathbf{R})\rangle$. This can be obtained from (5) by inserting the adiabatic approximate wavefunction as given in (1). We have denoted the adiabatic open path Berry phase as $\gamma_n(\Gamma)$ to distinguish from the more general geometric phase Φ_g. Thus, the adiabatic Berry phase is nothing but the line integral of a generalised gauge potential $\Omega_n(\mathbf{R}) = i\langle\chi_n(\mathbf{R}) \mid \nabla\chi_n(R)\rangle$ over the parameter space. The relation between this gauge potential and Berry potential can be worked out and it follows that

$$\Omega_n(\mathbf{R}) = \mathbf{A}_n(\mathbf{R}) - \mathbf{P}_n(\mathbf{R}), \tag{7}$$

where $\mathbf{P}_n(\mathbf{R})$ is a new gauge potential, given by

$$\begin{aligned}\mathbf{P}_n(\mathbf{R}) = \frac{i}{2\,|\langle\Psi_n(\mathbf{R}(0)) \mid \Psi_n(\mathbf{R})\rangle|^2} [\langle\Psi_n(\mathbf{R}(0))\,|\,(|\nabla\Psi_n(\mathbf{R})\rangle\langle\Psi_n(\mathbf{R})| \\ - |\Psi_n(\mathbf{R})\rangle\langle\nabla\Psi_n(\mathbf{R})|)\,|\Psi_n(\mathbf{R}(0))\rangle].\end{aligned} \tag{8}$$

By virtue of its transformation property under a local gauge transformation one can make sure that $\mathbf{P}_n(\mathbf{R})$ is a vector potential in the parameter space (see below).

Thus, like the Berry potential $\mathbf{A}_n(\mathbf{R})$, $\mathbf{P}_n(\mathbf{R})$ is a vector potential defined over the whole parameter space except that the latter depends on the initial point of the curve. For example, if we change the initial value of the parameter the value of the gauge potential will be different. In fact, this property of the gauge potential $\mathbf{P}_n(\mathbf{R})$ ensures the non-integrable nature of the open-path Berry phase. Now the open-path Berry phase can be given a *gauge theoretic description* in terms of these potentials as

$$\gamma_n(\Gamma) = \int_{\mathbf{R}(0)}^{\mathbf{R}(t_f)} [\mathbf{A}_n(\mathbf{R}) - \mathbf{P}_n(\mathbf{R})] \cdot d\mathbf{R}, \tag{9}$$

which says that the open-path Berry phase is the line integral of the difference of these two potentials in the parameter space.

This phase has the following properties. It is real, because both the potentials are real. It is independent of the parameter that we use to parametrise the evolution curve. It does not depend explicitly on the Hamiltonian or eigenvalue of the system. It is non-additive in nature which in turn attributes a memory to the adiabatically evolving quantal state. Hence, it qualifies to be called as the Berry phase for open-paths in parameter space. One can check that in the limiting case, the open-path Berry phase formula obtained by us precisely goes over to the cyclic Berry phase when the parameters come back to their original value after some time $t_f = T$.

Next we explicitly show the invariance of the open-path Berry phase under gauge and phase transformations. Under $U(1)$ local gauge transformation of the adiabatic eigenstate $|\Psi_n(\mathbf{R})\rangle$, we have $|\Psi_n(\mathbf{R}) \to e^{i\alpha(\mathbf{R})}\,|\Psi_n(\mathbf{R})\rangle$. It induces a gauge transformations on $\mathbf{A}_n(\mathbf{R})$ as well as on $\mathbf{P}_n(\mathbf{R})$:

$$\begin{aligned} \mathbf{A}_n(\mathbf{R}) &\to \mathbf{A}_n(\mathbf{R}) - \nabla\alpha(\mathbf{R}) \\ \mathbf{P}_n(\mathbf{R}) &\to \mathbf{P}_n(\mathbf{R}) - \nabla\alpha(\mathbf{R}). \end{aligned} \tag{10}$$

Therefore, the open-path Berry phase is clearly gauge invariant, because under local gauge transformations these vector potentials transform in the same way and hence their difference is gauge-compensated.

Further, it can be shown that the open-path Berry phase is also invariant under phase transformations. On redefining the phases of the adibatic eigenstate as

$$|\Psi_n(\mathbf{R})\rangle \to |\Psi_n(\mathbf{R})\rangle \exp\left(i\int_0^{\mathbf{R}} \mathbf{K}(\mathbf{R}')\cdot d\mathbf{R}'\right), \tag{11}$$

we can see that it affects both the vector potentials. The Berry potential and the new potential undergo transformations as

$$\begin{aligned} \mathbf{A}_n(\mathbf{R}) &\to \mathbf{A}_n(\mathbf{R}) - \mathbf{K}(\mathbf{R}) \\ \mathbf{P}_n(\mathbf{R}) &\to \mathbf{P}_n(\mathbf{R}) - \mathbf{K}(\mathbf{R}). \end{aligned} \tag{12}$$

Therefore, the open-path Berry phase is unchanged under a phase transformation. These properties enables us to define the concept of Berry phase *even for an infinitesimal path* in the parameter space. For example, if the parameters are changed by an amount $\Delta\mathbf{R}$, the corresponding change in the Berry phase would be given by

$$\Delta\gamma_n = [\mathbf{A}_n(\mathbf{R}) - \mathbf{P}_n(\mathbf{R})] \cdot \Delta\mathbf{R}. \tag{13}$$

Here, some remarks concerning the gauge potential $\mathbf{P}_n(\mathbf{R})$ can be made as to whether it is a new geometric structure on the Hilbert space of quantum states. We will show that it is not only a *new geometric structure* but also can be regarded as a much *richer gauge structure* in the sense that the Berry potential is only a part of it. Indeed, we will show that it can be split into two parts: one is just the Berry potential and the other is related to the matrix elements of the product of projection operators and the force operator (the force operator is $-\nabla H(\mathbf{R})$. To see this explicitly, let us express $\mathbf{P}_n(\mathbf{R})$ as

$$\mathbf{P}_n(\mathbf{R}) = \frac{i}{2}\left[\frac{\langle \Psi_n(\mathbf{R}(0)) \mid \nabla\Psi_n(\mathbf{R})\rangle}{\langle \Psi_n(\mathbf{R}(\mathbf{0})) \mid \Psi_n(\mathbf{R}\rangle} - \frac{\langle \nabla\Psi_n(\mathbf{R}) \mid \Psi_n(\mathbf{R}(0))\rangle}{\langle \Psi_n(\mathbf{R}) \mid \Psi_n(\mathbf{R}(0))\rangle}\right]. \tag{14}$$

On inserting a complete set of eigenstates at parameter value $\mathbf{R}$, we have

$$\mathbf{P}_n(\mathbf{R}) = \mathbf{A}_n(\mathbf{R}) - \mathrm{Im} \sum_{m \neq n} \frac{\langle \Psi_n(\mathbf{R}(0)) \mid \Psi_m(\mathbf{R})\rangle}{\langle \Psi_n(\mathbf{R}(0)) \mid \Psi_n(\mathbf{R})\rangle} \frac{\langle \Psi_m(\mathbf{R})\mid \nabla H \mid \Psi_n(\mathbf{R})\rangle}{(E_n(\mathbf{R}) - E_m(\mathbf{R}))}, \tag{15}$$

where we have used the fact that $\mathbf{A}_n = -\mathrm{Im}\langle \Psi_n \mid \nabla\Psi_n\rangle$. The above expression clearly shows the richness of the new gauge structure and brings out the fact that the Berry potential is only a part of it. Also, it provides a suitable formula for the open-path Berry phase as

$$\gamma_n((\Gamma) = \int_\Gamma \mathrm{Im} \sum_{m \neq n} \frac{\langle \Psi_n(\mathbf{R}(0)) \mid \Psi_m(\mathbf{R})\rangle}{\langle \Psi_n(\mathbf{R}(0)) \mid \Psi_n(\mathbf{R})\rangle} \frac{\langle \Psi_m(\mathbf{R})\mid \nabla H \mid \Psi_n(\mathbf{R})\rangle}{(E_n(\mathbf{R}) - E_m(\mathbf{R}))} \cdot d\mathbf{R}, \tag{16}$$

which clearly shows the independence of the choice of the phase of the eigenstates. (One may recall the expression for the field strength $\mathbf{V}_n$ which was provided in the original paper of Berry [2] and note the similarity here.) The formula (16) is very useful and has been recently studied in connection with linear response theory of adiabatic quantum systems and in understanding the damping of collective excitations in Fermi systems [16], where the dynamics is chaotic. Also, this generalised Berry phase theory has been applied to physical systems (like the collection of electrons and nuclei) where one applies the Born–Openheimer approximation and it is found that the quantum fluctuation in the generator of the parameter change is related to the time correlation function of the "fast" system [17], thus establishing a fluctuation-correlation theorem in the many-body context. The connection between the quantum metric tensor, force-force correlation, and the open-path Berry phase has been discussed for integrable and chaotic quantum systems.

3. CONNECTION BETWEEN THE HANNAY ANGLE AND BERRY PHASE USING COHERENT STATES

Consider the classical counterpart of the quantum system with N degrees of freedom, where the Hamiltonian of the system is given by $H(\mathbf{q}, \mathbf{p}, \mathbf{R})$. We assume that there exist N constants of motion in involution and the dynamical system is thus integrable. The classical trajectories are confined to the N-dimensional manifold, which is an N-dimensional torus. It is known that for integrable systems, we can go over to the action-angle (I_i, θ_i), $i=1, 2, ..., N$, description where the actions remain invariant during an adiabatic excursion. The angle variables undergo additional shift (Hannay angle) during a cyclic variation of parameters. The total change in the angular coordinate of the trajectory in phase space is thus given by

$$\theta_i(T) = \theta_i(0) + \int_o^T \varphi_i(\mathbf{I}, \mathbf{R}(\mathbf{t}))\, dt + \oint \left\langle \frac{\partial \theta_i}{\partial \mathbf{R}} \right\rangle \cdot d\mathbf{R}. \tag{17}$$

The above expression consists of a dynamical angle shift (given by the time integral of the instantaneous frequency) and a geometric angle shift, the latter being known as the Hannay angle [3]. Like the Berry connection which does not depend on the precise form of the Hamiltonian but only on its symmetries, similarly the Hannay one-form depends on the symmetries of the classical Hamiltonian. The symmetries in this case are the canonical automorphism of the invariant tori in phase space [18]. The standard formula for the Hannay angle, however, is not valid if the parameters are not brought back to their original value because under a rotation with respect to the angle variables of the phase space trajectories, the Hannay angle does not remain invariant. Remembering the difficulties encountered in this problem, which we have mentioned in the Introduction, it is natural to look for the connection between the open-path Berry phase and geometrical angle shift.

Here, we bring out the connection between the phase holonomy and angle holonomy using the parametrised coherent state formalism that describes the action and angle variables in the classical limit. In the sequel, we closely follow the methods of Maamache *et al.* [19]. For simplicity, let us restrict ourselves first to one degree of freedom. Given an adiabatically changing Hamiltonian $H(\mathbf{R})$ we can define a coherent state for the quantum system as

$$|\alpha, \mathbf{R}\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |\Psi_n(\mathbf{R})\rangle. \tag{18}$$

We can also define an excitation operator or quantum counting operator $N(\mathbf{R})$ as

$$N(\mathbf{R}) = \sum_{n=0}^{\infty} n\, |\Psi_n(\mathbf{R})\rangle\langle \Psi_n(\mathbf{R})| \tag{19}$$

and $N(\mathbf{R})$ satisfies an eigenvalue equation

$$N(\mathbf{R})\,|\Psi_n(\mathbf{R})\rangle = n\,|\Psi_n(\mathbf{R})\rangle. \tag{20}$$

In the classical limit ($h \to 0$, $n \to \infty$) the action is related to the excitation number n as $I = n\hbar$, which is finite. The coherent state is best suited for studying the classical limit as it represents a point in the phase space. The evolution of the coherent state represents the trajectory along which the actions remain invariant. Quantum mechanically, $|\alpha|^2$ represents the average value of the counting operator and in the classical limit $\hbar\,|\alpha|^2$ represents invariant action. Physically it has been argued [19] that the complex parameter $|\alpha(t)$ is related to the action and angle variable of the system as

$$\alpha(t) = \sqrt{\frac{I}{\hbar}}\, e^{-i\theta(t)}. \tag{21}$$

We can also express the adiabatic eigenstate in terms of action-angle state using the over completeness property of the coherent state. Since

$$|\Psi_n(\mathbf{R}(t))\rangle = \frac{1}{\pi}\int d^2\alpha e^{-|\alpha|^2/2}\, \frac{\alpha^{*n}}{\sqrt{n!}}\,|\alpha, \mathbf{R}\rangle, \tag{22}$$

where $d^2\alpha = d(\mathrm{Re}\,\alpha)\, d(\mathrm{Im}\,\alpha) = 1/(2\hbar)\, dI\, d\theta$, we can express the correspondence between the quantum eigenstate and a point in phase space parametrised by the action and angle variable as

$$|\Psi_n(\mathbf{R}(t))\rangle = \frac{1}{2\pi\hbar^{(n/2+1)}}\int dI\, d\theta e^{-I^2/2\hbar^2}\, I^{n/2}\, \frac{e^{-in\theta}}{\sqrt{n!}}\,|I, \theta, \mathbf{R}\rangle, \tag{23}$$

where we have denoted $|\alpha, \mathbf{R}\rangle = |I, \theta, \mathbf{R}\rangle$.

As the system evolves from some parameter value $\mathbf{R}(0)$, the classical trajectory starts from some initial angle coordinate on the constant action surface. We wish to compute what would be the angle shift for some arbitrary parameter value $\mathbf{R}(t_f)$. Quantally, consider the evolution of the initial coherent state $|\alpha(0), \mathbf{R}(0)\rangle$. Then, at a later time t, the state is given by

$$|\alpha(t), \mathbf{R}(t)\rangle = U(t)\,|\alpha(0), \mathbf{R}(0)\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\, e^{i\Phi_n(t)}\,|\Psi_n(\mathbf{R}(t))\rangle, \tag{24}$$

where we have used the fact that $U(t)\,|\Psi_n(\mathbf{R}(0))\rangle = e^{(i\delta_n(t) + i\gamma_n(t))}\,|\Psi_n(\mathbf{R}(t))\rangle = e^{i\Phi_n(t)}\,|\Psi_n(\mathbf{R}(t))\rangle$. Since, in the classical limit, the sum over n is highly peaked arround the value $N = |\alpha|^2$, most of the contribution to the sum comes from $n = N$. With this idea, we can expand $\Phi_n(t)$ to first order in $(n - N)$

$$\Phi_n(t) = \Phi_N(t) + (n - N)\,\frac{\partial \Phi_N(t)}{\partial N}. \tag{25}$$

Now, the parametrised coherent state at a later time t is given by

$$|\alpha(t), \mathbf{R}(t)\rangle = e^{i(\Phi_N(t) - N(\partial \Phi_N(t)/\partial N)}\,|\alpha(0)\,e^{i(\partial \Phi_N(t)/\partial N)}, \mathbf{R}(t)\rangle. \tag{26}$$

To know the angle shift during a non-cyclic variation of external parameters, we take the inner product of the initial and final (at time $t = t_f$) coherent state, which is given by

$$\langle \alpha(0), \mathbf{R}(0)\,|\,\alpha(t_f), \mathbf{R}(t_f)\rangle = e^{i(\Phi_N(t_f) - N(\partial \Phi_N(t_f)/\partial N))}\,e^{-|\alpha|^2} \sum_{n,\,m} \frac{\alpha(0)*^n}{\sqrt{n!}}\frac{\alpha(0)^m}{\sqrt{m!}} \times e^{im(\partial \Phi_N(t_f)/\partial N)}\,\langle \Psi_n(\mathbf{R}(0))\,|\,\Psi_m(\mathbf{R}(t_f))\rangle. \tag{27}$$

Using the random. phase approximation, one can neglect terms $n \neq m$ and thus the above expression reduces to

$$\langle \alpha(0), \mathbf{R}(0)\,|\,\alpha(t_f), \mathbf{R}(t_f)\rangle = e^{i(\Phi_N(t_f) - N(\partial \Phi_N(t_f)/\partial N))}\,e^{-|\alpha|^2} \sum_{n} \frac{|\alpha|^{2n}}{n!} \times e^{in(\partial \Phi_N(t_f)/\partial N)} e^{i\beta_n(t_f)}\,|\langle \Psi_n(\mathbf{R}(0))\,|\,\Psi_n(\mathbf{R}(t_f))\rangle|, \tag{28}$$

where $\beta_n(t_f) = \int_{\mathbf{R}(0)}^{\mathbf{R}(t_f)} \mathbf{P}_n(\mathbf{R}) \cdot d\mathbf{R}$. Following a similar argument as above, we replace the phase $\beta_n(t_f)$ in the classical limit to its first order approximation, viz., $\beta_n(t_f) = \beta_N(t_f) + (n - N)(\partial \beta_N(t_f)/\partial N)$. Therefore, the inner product between the initial and final coherent state is given by

$$\begin{aligned}\langle \alpha(0), \mathbf{R}(0)\,|\,\alpha(t_f), \mathbf{R}(t_f)\rangle &= e^{i(\Phi_N(t_f) - N(\partial \Phi_N(t_f)/\partial N))} e^{i(\beta_N(t_f) - N(\partial \beta_N(t_f)/\partial N))} \\ &\quad \times e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!}\, e^{in(\partial \Phi_N(t_f)/\partial N) + (\partial \beta_N(t_f)/\partial N))}\,|\langle \Psi_n(\mathbf{R}(0))\,|\,\Psi_n(\mathbf{R}(t_f))\rangle|.\end{aligned} \tag{29}$$

The phase factors appearing outside the summation are just the global phase factors and do not contribute to the relative phase shift of the adiabatic eigenstate, which would correspond in the classical limit to the relative angle shift. Therefore, the total angle shift would be given by the terms that appear inside the summation, i.e.,

$$\theta(t_f) - \theta(0) = \Delta\theta = -\frac{\partial \Phi_N(t_f)}{\partial N} + \frac{\partial \beta_N(t_f)}{\partial N} = -\frac{\partial \delta_N(t_f)}{\partial N} - \frac{\partial \gamma(\Gamma)}{\partial N}, \tag{30}$$

where the first term is the usual dynamical angle shift and the second term $\partial\gamma_N(\Gamma)/\partial N = \partial/\partial N(\gamma_N(t_f) - \beta_N(t_f))$ is the geometrical angle shift or Hannay angle for open-path excursions of the parameters. Therefore, the connection between the Hannay angle and Berry phase in the classical limit is given by

$$\theta(I, \Gamma) = -\hbar \frac{\partial\gamma(I, \Gamma)}{\partial I}. \tag{31}$$

For N-degrees of freedom, the system admits I_i and θ_i, $i = 1, 2, ..., N$, action and angle variables, respectively. It is straightforward to generalise the connection between the Hannay angle and Berry phase using product coherent states $\Pi_i\, |\alpha_i(t), \mathbf{R}(\mathbf{t})\rangle$, where each $\alpha_i(t)$ describes the I_ith action and θ_i angle variable. When the parameters follow a non-cylic variation, then each angle variable θ_i undergoes an additional shift given by

$$\theta_i(\mathbf{I}, \Gamma) = -\hbar \frac{\partial\gamma(\mathbf{I}, \Gamma)}{\partial I_i}. \tag{32}$$

4. SEMICLASSICAL LIMIT AND HANNAY ANGLE

In the foregoing discussions we describe how to obtain the semiclassical Berry phase and the Hannay angle for open-path excursions in parameter space. Berry [12] has analysed his closed-path phase in the semiclassical limit and established a connection to the classical Hannay angle. In the same spirit one can analyse the open-path Berry phase and derive the expression for adiabatic angle holonomy for open-path excursions of the classical Hamiltonian. In the semiclassical analysis, it is assumed that the eigenfunction is associated with a torus and the actions are quantised according to the Bohr–Sommerfeld [20] rule. The semiclassical expression for the wavefunction [12] is

$$\Psi_n(\mathbf{q}; \mathbf{R})\rangle = \langle \mathbf{q} | n(\mathbf{R})\rangle = \sum_\alpha a_{(\alpha)}(\mathbf{q}, I; \mathbf{R}) \exp\left(\frac{i}{\hbar} S^{(\alpha)}(\mathbf{q}, I; \mathbf{R})\right), \tag{33}$$

where the amplitude $a^2_{(\alpha)} = (1/(2\pi)^N)(d\theta^{(\alpha)}/d\mathbf{q}) = (1/(2\pi)^N) \det(\partial\theta_i^{(\alpha)}/\partial q_i)$ and α labels different branches of the multivalued, classical generating function $S^{(\alpha)}(\mathbf{q}, I; \mathbf{R})$. Each of the actions $S^{(\alpha)}$ satisfy the Hamilton–Jacobi equation. The existence of an invariant Lagrangian surface (torus) is important on which the multivalued actions $S^{(\alpha)}$ are defined. Using this wavefunction it is interesting to get the semiclassical Berry phase for open paths. Upon substitution, one will have two terms viz., the Berry term and the new term. The Berry potential can be easily evaluated and is given by

$$\mathbf{A}_n(\mathbf{R}) = -1/\hbar \int d\mathbf{q} \sum_\alpha \frac{1}{(2\pi)^N} \frac{d\theta^{(\alpha)}}{dq} \nabla S^{(\alpha)}(\mathbf{q}, I; \mathbf{R}), \tag{34}$$

where $\int d\mathbf{q} = \prod_{j=1}^{N} \int_{-\infty}^{+\infty} dq_j$ and in evaluating this, it is assumed that products of terms from different branches of α do not contribute because they give rise to rapid oscillations and cancel semiclassically on integrating over q. The additional term is not so straightforward to evaluate. However, we provide the closest simplified expression for it. Note that the vector potential $\mathbf{P}_n(\mathbf{R})$ can be written as

$$\mathbf{P}_n(\mathbf{R}) = -\mathrm{Im}\,\frac{\langle \Psi_n(\mathbf{R}(0)) \mid \nabla \Psi_n(\mathbf{R}(t))\rangle}{\langle \Psi_n(\mathbf{R}(0)) \mid \Psi_n(\mathbf{R}(t))\rangle}. \tag{35}$$

Within the semiclassical approximation this can be expressed as

$$\mathbf{P}_n(\mathbf{R}) = \frac{X(I;\mathbf{R})\,\nabla Y(I;\mathbf{R}) - Y(I;\mathbf{R})\,\nabla X(I;\mathbf{R})}{(X(I;\mathbf{R})^2 + Y(I;\mathbf{R})^2)}, \tag{36}$$

where

$$X(I;\mathbf{R}) = \int d\mathbf{q} \sum_{\alpha} a_{(\alpha)}(\mathbf{q}, I; \mathbf{R}(0))\, a_{(\alpha)}(\mathbf{q}, I; \mathbf{R}) \times \cos\left[\frac{1}{\hbar}(S^{(\alpha)}(\mathbf{q}, I; \mathbf{R}) - S^{(\alpha)}(\mathbf{q}, I; \mathbf{R}(0))\right] \tag{37}$$

and

$$Y(I;\mathbf{R}) = \int d\mathbf{q} \sum_{\alpha} a_{(\alpha)}(\mathbf{q}, I; \mathbf{R}(0))\, a_{(\alpha)}(\mathbf{q}, I; \mathbf{R}) \times \sin\left[\frac{1}{\hbar}(S^{(\alpha)}(\mathbf{q}, I; \mathbf{R} - S^{(\alpha)}(\mathbf{q}, I; \mathbf{R}(0))\right]. \tag{38}$$

Here, also those terms in the above expression survive which come from the product of the same branches of α. Thus, the semiclassical Berry phase formula for the open path excursion in parameter space is given by

$$\gamma_n(\Gamma) = -\int_{\mathbf{R}(0)}^{\mathbf{R}(t_f)} \left[\frac{1}{\hbar}\int d\mathbf{q} \sum_{\alpha} \frac{1}{(2\pi)^N} \frac{d\theta^{(\alpha)}}{dq} \nabla S^{(\alpha)}(\mathbf{q}, I; \mathbf{R}) + \frac{X(I;\mathbf{R})\,\nabla Y(I;\mathbf{R}) - Y(I;\mathbf{R})\,\nabla X(I;\mathbf{R})}{(X(I;\mathbf{R})^2 + Y(I;\mathbf{R})^2)}\right] \cdot d\mathbf{R}. \tag{39}$$

In a simplified notation, the above formula can be expressed as

$$\gamma_n(\Gamma) = -\int \left[\frac{1}{\hbar}\langle \nabla S^{(\alpha)}\rangle + \frac{X\nabla Y - Y\nabla X}{(X^2 + Y^2)}\right] \cdot d\mathbf{R}, \tag{40}$$

where the integral over $\mathbf{q}$ has been converted to an integral over the angles using the Jacobians $a_{(\alpha)}^2$ and the limits of the integration is supressed because the curve in parameter space is arbitrary. Unless otherwise stated, the above limit is understood as starting from some initial value to a final value of the parameters. This is the semiclassical limit of the open-path Berry phase.

During an adiabatic transport arround a closed-circuit, the above expression reduces to that of the well known result of Berry [12]. When the time t_f is so choosen that $\mathbf{R}(t_f)=\mathbf{R}(T)=\mathbf{R}(0)$, then the last term does not contribute to the semiclassical geometric phase, i.e., the closed line-integral over the parameters gives us

$$\oint \left[\frac{X \nabla Y - Y \nabla X}{(X^2+Y^2)} \right] \cdot d\mathbf{R} = 0, \qquad \bmod 2\pi n \tag{41}$$

and hence the closed-circuit Berry phase for a loop C is given by

$$\gamma_n(C) = \iint_{\partial A = C} \mathbf{V}_n(\mathbf{R}) \cdot d\mathbf{S} \tag{42}$$

with $\mathbf{V}_n(\mathbf{R}) = (1/\hbar)\, \nabla \wedge \int \langle \nabla S^{(\alpha)} \rangle \cdot d\mathbf{R}$.

Next, we obtain the Hannay angle for adiabatically evolving systems around an open circuit that was promised in the beginning of this paper. Using the connection between the quantal geometric phase and the classical Hannay angle one can express the latter as

$$\theta_H(I; \Gamma) = -h\, \frac{\partial \gamma_n(\Gamma)}{\partial I}. \tag{43}$$

Therefore, the classical angle holonomy during the adiabatic variation of the Hamiltonian along an arbitrary path in parameter space connecting the points $\mathbf{R}(0)$ and $\mathbf{R}(t_f)$ is given by

$$\Delta\theta_H(I; \Gamma) = \frac{\partial}{\partial I} \int \langle \nabla S^{(\alpha)} \rangle \cdot d\mathbf{R} + \frac{\hbar}{(X_f^2+Y_f^2)} \left(X_f \frac{\partial Y_f}{\partial I} - Y_f \frac{\partial X_f}{\partial I} \right), \tag{44}$$

where $X_f = X(I; \mathbf{R}(t_f))$ and $Y_f = Y(I; \mathbf{R}(t_f))$.

Thus, the adiabatic system admits a Hannay angle for *for an open-path* which is the semiclassical limit of the quantal adiabatic phase. *The open-path Berry phase and its relation to the Hannay angle constitute the central results of these last sections.* The original Hannay angle (for closed-paths) is invariant under parameter-dependent and action-dependent transformations of the origin from which the angle θ's are measured. Here, the generalised Hannay angle will remain invariant under a more general type of parameter-dependent and action-dependent transformations. The additional term takes care of the invariance of the Hannay angle under arbitrary transformations. At the quantal level, this property corresponds to the invariance of the open-path Berry phase under parameter-dependent phase transformations of the eigenfunctions.

5. CLASSICAL LIMIT OF THE OPEN-PATH BERRY PHASE AND CONNECTION TO THE HANNAY ANGLE

5a. *Berry Phase from Instantaneous Projectors*

In this section, we intend to obtain the classical limit of the Berry phase when the parameters need not follow a cyclic evolution. Essentially, the problem reduces to finding the classical limit of the generalised one-form $\Omega_n^{(1)}$ or the vector potnetial $\Omega_n(\mathbf{R})$, so that one may be able to shed some light on what would be the classical angle holonomy for non-cyclic variations. To this end, we express the open-path Berry phase in terms of the averages of the commutators of the instantaneous projection operators $P_n(\mathbf{R}) = |\Psi_n(\mathbf{R})\rangle\langle\Psi_n(\mathbf{R})|$ as it will facilitate the classical limit with ease. This one-dimensional projection operator depends on the parameter continuously and undergoes a continuous evolution in parameter space. Since we are dealing with non-cyclic evolutions of parameters, $P_n(\mathbf{R}(t_f))$ is not equal to $P_n(\mathbf{R}(0))$. To express the Berry phase in terms of these projectors, note that (5) can be written as

$$\gamma_n(\Gamma) = \frac{i}{2}\int_\Gamma \Big(\langle\chi_n(\mathbf{R})\,|\,\nabla\chi_n(\mathbf{R})\rangle - \langle\nabla\chi_n(\mathbf{R})\,|\,\chi_n(\mathbf{R})\rangle\Big)\cdot d\mathbf{R}. \tag{45}$$

By expressing the "reference-eigenstate" $|\chi_n(\mathbf{R})\rangle$ as $|\chi_n(\mathbf{R})\rangle = (P_n(\mathbf{R})\,|\,\Psi_n(\mathbf{R}(0))\rangle)/(|\langle\Psi_n(\mathbf{R})\,|\,\Psi_n(\mathbf{R}(0))\rangle|)$, we have

$$\begin{aligned}\langle\chi_n(\mathbf{R})\,|\,\nabla\chi_n(\mathbf{R})\rangle &= \frac{\langle\Psi_n(\mathbf{R}(0))|\;P_n(\mathbf{R})\,\nabla P_n(\mathbf{R})\;|\Psi_n(\mathbf{R}(0))\rangle}{\langle\Psi_n(\mathbf{R}|\;P_n(\mathbf{R})\;|\Psi_n(\mathbf{R}(0))\rangle} \\ &\quad -\frac{1}{2}\frac{\langle\Psi_n(\mathbf{R}(0))|\;\nabla P_n(\mathbf{R})\;|\Psi_n(\mathbf{R}(0))\rangle}{\langle\Psi_n(\mathbf{R})|\;P_n(\mathbf{R})\;|\Psi_n(\mathbf{R}(0))\rangle}.\end{aligned} \tag{46}$$

Inserting the above equation into the geometric phase formula, we can write the open-path Berry phase in terms of the commutator of the projector and its gradient over the space of parameters, as is given by

$$\gamma_n(\Gamma) = \frac{i}{2}\int \frac{\langle\Psi_n(\mathbf{R}(0))|\;[P_n(\mathbf{R}), \nabla P_n(\mathbf{R})]\;|\Psi_n(\mathbf{R}(0))\rangle}{\langle\Psi_n(\mathbf{R})|\;P_n(\mathbf{R})\;|\Psi - n(\mathbf{R}(0))\rangle}\cdot d\mathbf{R}. \tag{47}$$

Thus, the generalised phase one-form would be given by

$$\Omega_n^{(1)} = \frac{i}{2}\frac{\langle\Psi_n(\mathbf{R}(0))|\;[P_n(\mathbf{R}), dP_n(\mathbf{R})]\;|\Psi_n(\mathbf{R}(0))\rangle}{\langle\Psi_n(\mathbf{R})|\;P_n(\mathbf{R})\;|\Psi_n(\mathbf{R}(0))\rangle}, \tag{48}$$

where d is the exterior derivative with respect to the parameters. A similar formula has been derived by Mead [22] for the case of cyclic evolutions and by Wagh [23] for non-cyclic evolutions in the projective Hilbert space of the quantum system after the present author introduced the concept of "reference-state." It is interesting to remark that the open-path Berry phase has its origin in the non-commutativity of the instantaneous projection operator with its exterior derivative in the parameter space, which is purely quantum mechanical in nature. This expression is more suitable to study the classical limit because there is a direct correspondence between the quantum mechanical commutator of hermitian operators and the classical valued Poisson bracket.

5b. *Classical Limit of the Berry Phase*

To analyse the classical limit of the open-path Berry phase we use the Wigner–Weyl representation of quantal expression and take the lowest order term (in powers of $\hbar$) that will correspond to the classical limit of the former. In Wigner representation [24] the quantum mechanical operator $\hat{O}$ is representated as a phase space function $O_W(\mathbf{q}, \mathbf{p})$, where

$$O_W(\mathbf{q}, \mathbf{p}) = \int d^N\mathbf{y} \langle \mathbf{q} + \mathbf{y}/2|\, \hat{O}\, |\mathbf{q} - \mathbf{y}/2\rangle\, e^{-i\mathbf{p}\cdot\mathbf{y}/\hbar}. \tag{49}$$

The Weyl symbol of the operator reduces to the classical valued function in the $\hbar \to 0$ limit. If we choose $\hat{O}$ to be a density operator $\hat{\rho} = |\Psi\rangle\langle\Psi|$ constructed from a pure state wavefunction, then we get the Wigner function

$$\rho_W(\mathbf{q}, \mathbf{p}) = \int d^N\mathbf{y} \rho(\mathbf{q} - \mathbf{y}/2, \mathbf{q} + \mathbf{y}/2)\, e^{-i\mathbf{p}\cdot\mathbf{y}/\hbar}. \tag{50}$$

Wigner representation of the phase space density and phase space function is an alternate approach to ordinary quantum mechanics where one can talk of the classical limit of various quantities with ease. In this representation, we can express the average of the commutator as a phase space average of the Weyl symbol of the commutator between the projection operators, i.e.,

$$\begin{aligned} &\langle \Psi_n(\mathbf{R}(0))|\, [P_n(\mathbf{R}), \nabla P_n(\mathbf{R})]\, |\Psi_n(\mathbf{R}(0))\rangle \\ &\quad = \int d^N\mathbf{q} d^N\mathbf{p} P_n(\mathbf{q}, \mathbf{p})([P_n(\mathbf{R}), \nabla P_n(\mathbf{R})])_W(\mathbf{q}, \mathbf{p}) \end{aligned} \tag{51}$$

and similarly, we have for the denominator

$$\langle \Psi_n(\mathbf{R}(0))|\, P_n(\mathbf{R})\, |\Psi_n(\mathbf{R}(0))\rangle = \int d^N\mathbf{q} d^N\mathbf{p} P_n(\mathbf{q}, \mathbf{p})\, P_n(\mathbf{q}, \mathbf{p}, \mathbf{R}). \tag{52}$$

The Weyl symbol of the commutator is given in terms of the Moyal bracket

$$([P_n(\mathbf{R}), \nabla P_n(\mathbf{R})])_W = \frac{2}{i} P_n(\mathbf{q}, \mathbf{p}, \mathbf{R}) \sin\sigma\, \nabla P_n(\mathbf{q}, \mathbf{p}, \mathbf{R}). \tag{53}$$

where σ is given by

$$\sigma = \sum_{i=1}^{N} \frac{\hbar}{2}\left(\frac{\partial^{\leftarrow}}{\partial \mathbf{p}} \cdot \frac{\partial^{\rightarrow}}{\partial \mathbf{q}} - \frac{\partial^{\leftarrow}}{\partial \mathbf{q}} \cdot \frac{\partial^{\rightarrow}}{\partial \mathbf{p}}\right), \tag{54}$$

where the left and right arrows on the differential operators imply that they act on the functions which lie to the left and right, respectively. Since we are interested only in the classical limit of the generalised vector potential, the Weyl symbol of the commutator goes over to the poisson bracket of the corresponding distribution functions on phase space. Hence, we have

$$([P_n(\mathbf{R}), \nabla P_n(\mathbf{R})])_W \rightarrow \frac{1}{i}\{P_n(\mathbf{q}, \mathbf{p}, \mathbf{R}), \nabla P_n(\mathbf{q}, \mathbf{p}, \mathbf{R})\}_{P.B}. \tag{55}$$

Also, for an integrable system we know that the invariant manifold is the torus on which N actions remain constant and the initial phase space distribution can be taken as a microcanonical distribution, where $P(\mathbf{q}, \mathbf{p})$ is given by [25]

$$P_n(\mathbf{q}, \mathbf{p}) = \frac{1}{(2\pi)^N}\,\delta^N(\mathbf{I}(\mathbf{q}, \mathbf{p}) - \mathbf{I}). \tag{56}$$

This N-dimensional delta function tells us that the Wigner function for an eigenstate is concentrated in the region that a classical orbit visits over an infinite time. The phase space average of any function is defined as

$$\langle f \rangle_I = \frac{1}{(2\pi)^N} \int d^N\mathbf{q}\, d^N\mathbf{p} f(\mathbf{q}, \mathbf{p}, \mathbf{R})\, \delta^N(\mathbf{I}(\mathbf{q}, \mathbf{p}) - \mathbf{I}). \tag{57}$$

Therefore, the classical limit of the gegeneralised vector potential is given by

$$\Omega_c(\mathbf{R}) = \frac{-1/2 \int d^N\mathbf{q}\, d^N\mathbf{p}\, \delta^N(\mathbf{I}(\mathbf{q}, \mathbf{p}) - I) \cdot \{P(\mathbf{q}, \mathbf{p}, \mathbf{R}), \nabla P(\mathbf{q}, \mathbf{p}, \mathbf{R})\}_{P.B}}{\int d^N\mathbf{q}\, d^N\mathbf{p}\, \delta^N(\mathbf{I}(\mathbf{q}, \mathbf{p}) - \mathbf{I})\, P(\mathbf{q}, \mathbf{p}, \mathbf{R})} \tag{58}$$

Thus, the classical angle holonomy θ_H^c for integrable systems would be given by

$$\theta_H^c = \int \Omega_c(\mathbf{R}) \cdot d\mathbf{R} = -\frac{1}{2}\int \frac{\langle \{P(\mathbf{q}, \mathbf{p}, \mathbf{R}), \nabla P(\mathbf{q}, \mathbf{p}, \mathbf{R})\}_{P.B} \rangle_I}{\langle P(\mathbf{q}, \mathbf{p}, \mathbf{R}) \rangle_I} \cdot d\mathbf{R}, \tag{59}$$

which suggests that the origin of the angle holonomy could be due to the non-vanishing nature of the torus average of the phase space density with its gradient in parameter space. However, it is not at all clear to the author how to prove this statement purely using classical arguments.

5c. *Operational Definition of the Hannay Angle for Open-Paths*

Although it is difficult to derive the non-cyclic Hannay angle at classical level, we can try to give an operational definition of it. This would require the knowledge of the *classical analog* of the quantum mechanical inner product of any two vectors in the Hilbert space of the quantum system. In quantum theory the most important thing is the inner product between two non-orthogonal states which is in general a complex number. Physically, this represents the survival amplitude of a system in a certain state once it is prepared in a given initial state. Is there any such thing in the classical world? This is a question which bothers some physicist that I know and the answer is not quite clear. However, we can try to see what is the *classical limit* of the quantum mechanical inner product. It may be remarked that the square of the modulus of the inner product (transition probability) between two states can be expressed in terms of Wigner functions and in the classical limit this will represent the overlap integral of microcanonical distributions corresponding to two possible configurations.

Consider two quantum states $|\Psi_1\rangle = |\Psi(0)\rangle$ and $|\Psi_2\rangle = |\Psi(t)\rangle$ whose inner product is defined on the Hilbert space of the quantum system. If $U(t)$ is the unitary operator that generates $|\Psi_2\rangle$ from $|\Psi_1\rangle$, then the inner product can be expressed as

$$\begin{aligned}\langle \Psi_1 \mid \Psi_2\rangle &= \langle \Psi(0)|\, U(t)\, |\Psi(0)\rangle = tr(\rho(0)\, U(t)) \\ &= \int d^N\mathbf{q}\, d^N\mathbf{p}\, \rho_W(\mathbf{q}, \mathbf{p})\, U_W(\mathbf{q}, \mathbf{p}, t),\end{aligned} \tag{60}$$

which is nothing but the phase space average of the unitary operator over the Wigner distribution. The classical limit of this would correspond to the phase space average of the classical function that generates the canonical transformation. For adiabatic eigenstates let $U(\mathbf{R}(t_f)), \mathbf{R}(0))$ be the unitary operator that relates the states $|\Psi_n(\mathbf{R}(t_f))\rangle$ and $|\Psi_n(\mathbf{R}(0))\rangle$. Then the inner product between the initial and final adiabatic eigenstates can be written as an average of the unitary operator $U(\mathbf{R}(t_f)), \mathbf{R}(0))$. Thus, $\langle \Psi_n(\mathbf{R}(0)) \mid \Psi_n(\mathbf{R}(t_f))\rangle = \langle \Psi_n(\mathbf{R}(t_f)) \mid U(\mathbf{R}(t_f)), \mathbf{R}(0)) \times |\Psi_n(\mathbf{R}(t_f))\rangle$. Since any unitary operator can be written as $U = C + iS$, where C and S are commuting hermitian operators, the quantum mechanical inner product is given by $\langle C\rangle + i\langle S\rangle$, which is in general a complex number. We replace the quantum mechanical averages by its classical ones, where the averages of C and S are taken over microcanonical distributions and are given by

$$\langle C\rangle_{\mathbf{I}} = \frac{1}{(2\pi)^N}\int d^N\mathbf{q}\, d^N\mathbf{p} C(\mathbf{q}, \mathbf{p}, \mathbf{R}(t_f), \mathbf{R}(0))\, \delta^N(\mathbf{I}(\mathbf{q}, \mathbf{p}) - \mathbf{I}), \tag{61}$$

$$\langle S\rangle_{\mathbf{I}} = \frac{1}{(2\pi)^N}\int d^N\mathbf{q}\, d^N\mathbf{p} S(\mathbf{q}, \mathbf{p}, \mathbf{R}(t_f), \mathbf{R}(0))\, \delta^N(\mathbf{I}(\mathbf{q}, \mathbf{p}) - \mathbf{I}). \tag{62}$$

Here, as before the averages [3] are taken around the Hamiltonian contour on which the point $(\mathbf{q}, \mathbf{p})$ lies and they are functions of the action $\mathbf{I}$, initial and final parameter value. The quantities C_c, $(\mathbf{q}, \mathbf{p}, \mathbf{R}(t_f), \mathbf{R}(0))$ and $S_c(\mathbf{q}, \mathbf{p}, \mathbf{R}(t_f), \mathbf{R}(0))$ are classical valued functions whose Poisson bracket vanishes and is related to the generator of the canonical transformation in classical phase space. Therefore, one could write the classical analogue of the quantum mechanical inner product as $\langle C\rangle_{\mathbf{I}}+i\langle S\rangle_{\mathbf{I}}$. With this idea one can give an operational definition of the non-cyclic Hannay angle as

$$\theta_H(I;\Gamma)=\int\left\langle\frac{\partial\theta}{\partial\mathbf{R}}\right\rangle\cdot d\mathbf{R}+\tan^{-1}\left(\frac{\langle S\rangle_{\mathbf{I}}}{\langle C\rangle_{\mathbf{I}}}\right), \tag{63}$$

where the first term is the usual Hannay term and the second term represents an additional angle coming from the argument of the classical limit of the quantum mechanical inner product. In the future one may be able to derive the open-path Hannay angle within the classical mechanics–which seems to be a difficult task at present.

6. DISCUSSION AND CONCLUSION

In this section we discuss briefly an application of the open-path Berry phase and conclude the formalism that has been developed in this paper. The open-path Berry phase and its classical counterpart can have important applications in many areas of physics. Here, we will illustrate how it shows up in an interesting way for the case of wave-packet revivals. The revival phenomenon refers to the case where a quantal wave-packet spreads following a classical trajectory, reassembles after some time T_R (called the revival time), and then takes the course of the classical trajectory. This phenomenon [21] which was well studied for time-independent Hamiltonians, recently has been generalised by Jarzynski [10] to the case of adiabatically changing Hamiltonian systems. He has shown that if initially the quantal wave-packet is at some point (say) $(\mathbf{q}_0, \mathbf{p}_0)$ in phase space, then the effect of adiabatic changes of external parameters can be manifested as a displacement of the location of the revived wave-packet along its classical trajectory. The amount by which the packet is shifted is equal to the adiabatic, closed-circuit Hannay angle. In carrying out his analysis it is assumed that the external parameters are varied in a cyclic manner and the time period T over which the parameters return to their original value is just equal to the revival time T_R. He has concluded that the effect of the Berry phase on the revival phenomenon is meaningful only when the revival time T_R coincides with the cyclic time. As we have shown in this paper the Berry phase and Hannay angle are not only well defined for closed paths but also for open paths. It must be now evident that the *effect of the classical Berry phase on wave-Packet revival can be seen even when the parameters do not come back to their original value at time* T_R. Hence, we argue that the nice conclusion of Jarzynski need not be restrictive to the case considered by him, although his analysis may need a modification (to properly take into account the contributions coming from the new vector potential). If one probes the location of two identically prepared wave-packets during their evolution along the classical trajectory by keeping the parameters of one packet constant and varying the parameters of the other in any desirable way, one will be able to demonstrate the existence of the open-path Hannay angle in wave-packet revivals. By observing the relative shift in the locations of the revived packet one may infer the effect of the open-path Hannay angle.

To conclude this paper, in Section 2, we obtained the Berry phase for quantum (whose classical counterpart is integrable) systems when parameters follow an open path during an adiabatic evolution. The reason for such a motivation has been clearly brought out. It is found that a generalised gauge potential (quantum one-form) can be defined over the parameter space whose line integral gives the Berry phase for open-path excursions of the parameters. The open-path Berry phase is shown to be gauge invariant and also phase invariant. Further, the non-cyclic Berry phase goes over to the usual Berry phase formula for the cyclic path.

The classical angle holonomy for the open path is not known and there is no way to proceed because for non-cyclic variations of external parameters it is not clear how to compare the angle variables. In Section 3, we provided a connection between the open-path Berry phase and Hannay angle using parametrised coherent states, which describes action-angle variables appropriately. It is found that the open-path Hannay angle can be obtained by taking a partial derivative of the open-path Berry phase with respect to the quantum number in large n limit (classical limit).

In Section 4, using the semiclassical approximation for the wavefunction we evaluated the open-path Berry phase and subsequently derived the semi-classical Hannay angle. The open-path Hannay angle contains an extra term which is ususally absent for the cyclic angle holonomy of the integrable system. In Section 5, we analysed the classical limit of the quantum one-form by expressing it in terms of the commutator of the instantaneous projection operators with its exterior derivative. This enables us to take the classical limit by using the correspondence rule between the commutator and the Poisson bracket. Using the Wigner representation of the distribution function and its classical counterpart we expressed the angle holonomy in terms of the torus averages of the Poisson bracket of the phase space density with its exterior derivative. It may be argued that the quantum mechanical inner product has a classical limit which gives rise to an additional term in the Hannay angle for open path excursions. The operational definition of the non-cyclic Hannay angle is given within the classical mechanics–*whose derivation is still an open problem.* As an application we outlined how this angle holonomy can have an important effect in wave-packet revivals. The future challenge lies in establishing the open-path Hannay angle purely from classical considerations. Since not much is known about this interesting angle holonomy when the parameters do not follow a closed path, it is hoped that this work will be an important step in this direction.

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REFERENCES

1. B. Simon, *Phys. Rev. Lett.* **51** (1983), 2167.
2. M. V. Berry, *Proc. R. Soc. London A* **392** (1984), 457.
3. J. H. Hannay, *J. Phys. A* **18** (1985), 221.
4. Y. Aharonov and J. Anandan, *Phys. Rev. Lett.* **58** (1987), 1593.
5. J. Samuel and R. Bhandari, *Phys. Rev. Lett.* **60** (1988), 2339.
6. N. Mukunda and R. Simon, *Ann. Phys.* **228** (1993), 20.
7. A. K. Pati, *Phys. Rev. A* **52** (1995), 2576.
8. M. V. Berry and J. H. Hannay, *J. Phys. A* **21** (1988), L325.
9. J. Anandan, *Phys. Lett. A* **129** (1988), 201.
10. C. Jarzynski, *Phys. Rev. Lett.* **74** (1995), 1264.
11. E. Gozzi and W. D. Thacker, *Phys. Rev. D* **35** (1987), 2388, 2398.
12. M. V. Berry, *J. Phys. A* **18** (1985), 15.
13. Y. S. Wu and H. Z. Li, *Phys. Rev. B* **38** (1988), 11, 907.
14. S. Pancharatnam, *Proc. Indian Acad. Sci. A* **44** (1956), 247.
15. A. K. Pati, *J. Phys. A* **28** (1995), 2087.
16. S. R. Jain and A. K. Pati, *Phys. Rev. Lett.* **80** (1998), 650.
17. A. K. Pati, Fluctuation, Time-Correlation Function and Geometric Phase, preprint, 1997 (quant-ph/9804003).
18. S. Golin, A. Knauf, and S. Marmi, *Comm. Math. Phys.* **123** (1989), 95.
19. M. Maamache, J. P. Provost, and G. E. Valle'e, *J. Phys. A* **23** (1990), 5765.
20. J. B. Keller, *Ann. Phys. (N.Y.)* **4** (1958), 180 .
21. J. Parker and C. R. Stroud, Jr., *Phys. Rev. Lett.* **56** (1986), 716; I. Sh. Averbukh and N. F. Perelman, *Phys. Lett. A* **139** (1989), 449; M. Nauenberg, *J. Phys. B* **23** (1990), 385.
22. C. A. Mead, *Rev. Mod. Phys.* **64** (1992), 51.
23. A. G. Wagh, *J. Pure Appl. Phys.* **33** (1995), 566.
24. E. P. Wigner, *Phys. Rev.* **40** (1932), 749.
25. M. V. Berry, *Phil. Trans. R. Soc. London* **287** (1977), 237.