HARISH CHANDRA RESEARCH INSTITUTE

QUANTUM FIELD THEORY II

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Assignment # 3

Due - March 30

1. Let M be an even dimensional antisymmetric matrix. Show that there is a unitary matrix U such that

$$U^T M U = B,$$

where B is a block diagonal matrix with 2×2 diagonal blocks, with the diagonal blocks D_i having the form

$$D_i = \begin{pmatrix} 0 & d_i \\ -d_i & 0 \end{pmatrix} .$$

[10 points]

2. Let ξ_i, η_i be a set of (complex) Grassmann variables and $\bar{\xi}_i, \bar{\eta}_i$ be their complex conjugates. Let M be a complex matrix. Under what conditions on M is the quadratic form

$$+\bar{\xi}_i M_{ij} \xi_j + \bar{\eta}_i \xi_i + \bar{\xi}_i \eta_i$$

real? Perform the integral

$$\int d^n\xi d^n\bar{\xi} \exp\left(+\bar{\xi}_i M_{ij} \xi_i + \bar{\eta}_i \xi_i + \bar{\xi}_i \eta_i\right),$$

taking the quadratic form to be real.

[10 points]

3. By following steps similar to the previous problem, compute $Z(\eta(x), \bar{\eta}(x))$ for a free Dirac field coupled to an external current,

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi + \bar{\eta}\psi + \bar{\psi}\eta .$$

Recall that the definition of $\bar{\psi}$ is $\bar{\psi} \equiv \psi^\dagger \gamma^0$ ($\psi^\dagger$ is the complex conjugate of ψ). Give all details of the steps involved.

[20 points]

4. Consider a free scalar field $\varphi(x)$ of mass m in four dimension. Construct the conserved charges P^μ associated with translational invariance in terms of creation and annihilation operators (perform the necessary normal ordering operations). Compute

$$[P^\mu, \varphi(x)] \quad \text{and} \quad [P^\mu, P^\nu]$$

[20 points]

5. Consider the Fourier transform of the 1PI Green's functions in position space. Show that these are proportional to a "momentum conserving" delta function. Use this result to write down Feynman rules in terms of 1PI vertices and exact propagators.

[10 points]

6. In general the quantum effective $\Gamma(\varphi(x))$ action is a non-local functional of $\varphi(x)$. Assuming that the Fourier transform of the 1PI Green's functions are finite at zero momentum, show that for constant field configurations ($\varphi(x) = \varphi_0$), $\Gamma(\varphi)$ is equal to a function of φ_0 times the volume of spacetime. Comment on the validity of the assumption the presence of massless particles

[10 points]

7. Evaluate the quantum effective action $\Gamma(x(t))$ for a constant configuration, i.e $x(t) = x_0$ for a harmonic oscillator i.e

$$\mathcal{L} = \frac{1}{2}m(\dot{x})^2 - \frac{1}{2}m\omega^2 x^2$$

First express your answer in terms of x_0 and the determinant of an operator. Then make use of the fact that $\log \mathrm{DetA} = \mathrm{Tr} \log \mathrm{A}$ to express the answer in terms of x_0 and an integral (which depends on the parameters in the Lagrangian). Is the integral convergent or divergent ?

[20 points]

8. Consider a theory with N scalar fields ϕ_a ($a = 1..N$). The system is described a Lagrangian which has a symmetry corresponding to the global $SO(N)$ rotation of the fields. What symmetry does this imply for the 1PI effective action the theory ?

[10 points]

9. Consider two theories scalar field theories (a, b) whose actions are related by $S_a(\varphi(x)) = S_b(\varphi(x) + F(x))$. How are their 1PI effective actions related ?

[10 points]