Harish Chandra Research Institute

Quantum Field Theory II

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Assignment # 4
Due - 16 April

1. The first step in the systematic procedure to obtain expressions for the 1PI vertices in the φ^3 theory is to draw all Feynman diagrams contributing to the 1PI vertex which have the property that there is no sub diagram in the diagram which can be regarded as corrections to the propagator or the three point vertex. Such diagrams can be classified according to the number of vertices they have. For the 1PI irreducible vertex with four external legs $i\mathrm{V}_4(k_1, k_2, k_3, k_4)$ the minimum number of vertices is four and with four vertices there is only one such diagram (upto exchange of external momenta); the box diagram -

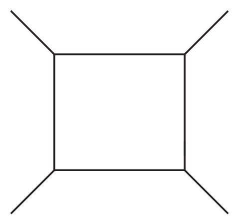

Figure 1: The Box Diagram

Draw all such diagrams which contribute to $i\mathrm{V}_4(k_1, k_2, k_3, k_4)$ (upto exchange of external momenta) with six vertices.

[10 points]

2. *Cut Off Regularisation:* The mathematical expressions for Feynman graphs after Wick rotation can be regulated by restricting the domain of integration in momentum space to a ball of radius Λ.

$$\int d^n p \; f(p) \xrightarrow{\text{Cut Off Regulate}} \int_{|p|<\Lambda} d^n p \; f(p) \tag{1}$$

Consider the phi-four theory in four dimensions.

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \varphi_0 \partial^\mu \varphi_0 - \frac{1}{2} m_0^2 \varphi_0^2 - \frac{1}{4!}\lambda_0 \varphi_0^4$$

Set up the perturbative expansion in the theory by writing

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{4!}\lambda \varphi^4 - \frac{1}{2} A \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} B \varphi^2 - \frac{1}{4!} C \varphi^4. \tag{2}$$

with φ normalised as $\langle k|\varphi(x)|0\rangle = e^{-ikx}$, m equal to the physical mass of the one particle excitations and $i\lambda = -i\Gamma^4_\varphi(p_1, p_2, p_3, p_4)_{s=4m^2, t=u=0}$ (s, t, u being the Mandelstam variables of the 1PI four vertex)

(a) Write down the Feynman rules for the Lagrangian (2) treating all terms other than the kinetic term and the $\frac{1}{2}m^2\varphi^2$ as vertices.

(b) Define the regulated theory to be the theory with the Feynman rules that you wrote in part (a) with divergent integrals evaluated using the prescription (1). Obtain the expressions for $\Pi_\varphi(k^2)$ and $-i\Gamma^4_\varphi(p_1,p_2,p_3,p_4)\big|_{s=4m^2,t=u=0}$ (s,t,u being the Mandelstam variables) at one loop in the regulated theory.

(c) Define quantities in the theory to be equal to the $\Lambda \to \infty$ of the the corresponding quantitates in the regulated theory. Hence determine the Λ dependence of A,B and C at one loop.

[5+15+5 points]

3. *Dimensional Regularisation:* A regulator should have the property that it modifies the integrands in loop integrals for very high values of the momenta (there by ensuring convergence) while keeping the the behaviour for low values unchanged. For dimensional regularisation (DR), this is not certainly obvious from the definition. Restricting ourselves to scalar theories at the one loop level, we will show in this problem that DR does indeed satisfy this property (the argument can be easily generalised to more general theories and higher loops). While dealing with scalar theories at the one loop level the integrals that require regularisation are of the form

$$I = \int dF \int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2+D)^b} \quad , \qquad b > 0$$

where dF in the measure for the integration of the Feynman parameters. In DR

$$I \xrightarrow{\text{DR}} I_\epsilon = \mu^\epsilon \int dF \int \frac{d^{n-\epsilon} q}{(2\pi)^{n-\epsilon}} \frac{1}{(q^2+D)^b}.$$

(a) To study the effect of DR on the n dimensional loop momenta one performs the integral only over the $-\epsilon$ dimensions. Perform this integral by writing $q^2 = p^2 + p^2_{-\epsilon}$ (p^2 and $p^2_{-\epsilon}$ are the square of the momentum in n and $-\epsilon$ dimensions; show that

$$I_\epsilon = \int dF \int \frac{d^n p}{(2\pi)^n} \frac{1}{(p^2+D)^b} \frac{\Gamma(b+\epsilon/2)}{\Gamma(b)} \left(\frac{4\pi\mu^2}{p^2+D}\right)^{\epsilon/2}$$

(b) Show that in the limit $\epsilon \to 0$ the integrand does not receive significant modification as long as

$$\log\left(\frac{p^2+D}{4\pi\mu^2}\right) \ll \frac{2}{\epsilon};$$

and that this implies that in DR the ultraviolet is modified at the scale $\Lambda^2_{\text{DR}} \approx 4\pi\mu^2 e^{2/\epsilon}$.

[10 + 5 points]

Note that the integrand can also get significantly modified in the infrared unless D is bounded from below. For massive theories the mass of the lightest particle provides such a bound; for theories with massless particles one has to be more careful while working with DR.

4. Problem 14.1 of Srednicki's textbook, where a generalisation of the Feynman's formula given in the lectures is discussed. [10 points]

5. Perform the integal

$$\int \frac{d^d q}{(2\pi)^d} \frac{|\vec{q}|^a}{(|\vec{q}|^2 + D)^b},$$

where $\vec{q}$ is a vector in d dimensional Euclidean space. Express your answer in terms of the Gamma function.

[10 points]