Workshop: Representation theory of Real groups and Automorphic forms

Venue: *Harish-Chandra Research Institute, Chhatnag Road, Jhunsi, Prayagraj -211019*

Date: October 02, 2023 - October 07, 2023

Title of the workshop: Representation theory of Real groups and Automorphic forms
Topic: Representation theory, Harmonic analysis and Number theory

A brief description of the Workshop:

In the workshop, we shall consider seven themes covering different aspects of the Representation theory of Real groups and Automorphic forms which would be followed up by an international conference on celebrating Harish-Chandra's centenary : (1) Compact Lie groups, their representations, Weyl Character formula. (2) Classification of all irreducible representations of $SL_2(\mathbb{R})$, Plancherel theorem for $SL_2(\mathbb{R})$. (3) Algebraic representation theory of real Lie groups through the introduction of $(\mathfrak{g}, K)$ modules. (4) Admissibility theorems, Character theory, introduction to Harish-Chandra's works on discrete series, and their characters. (5) Cohomological induction. (6) Notion of Automorphic representations due to Harish-Chandra. (7) Introduction to the Spectral decomposition of $L^2(G/\Gamma)$ due to Langlands.

Syllabus:

Name of the Speaker with affiliation	**Number of Lectures**	**Syllabus**
MS Raghunathan Professor CBS Mumbai	5 lectures of 1 hours each	Compact Lie groups, their representations, Weyl Character formula
EK Narayanan Professor IISc Bangalore	5 lectures of 1 hours each	Classification of all irreducible representations of $SL_2(\mathbb{R})$, Plancherel theorem for $SL_2(\mathbb{R})$
Dragan Milicic Professor Utah	5 lectures 1 hours each	Algebraic representation theory of real Lie groups through the introduction of $(\mathfrak{g}, K)$ modules. The Casselman-Wallach theorem on equivalence of categories Asymptotic of matrix coefficients. (Casselman-Milicic theorem)
R. Parthasarathy Professor Retd. from TIFR Mumbai	3 lectures of 1 hours each	Admissibility theorems, Character theory, Introduction to Harish-Chandra's works on discrete series, and their characters
Arvind Nair Professor TIFR Mumbai	3 lectures 1 hours each	Cohomological induction: Constructing and classifying all unitary representations of real reductive groups with cohomology, in particular, introducing the representations $A_{\mathfrak{q}}(\lambda)$ through the process of cohomological induction due to Zuckerman
Ravi Raghunathan Professor IIT Bombay	3 lectures of 1 hours each	Notion of Automorphic representations due to Harish-Chandra, finiteness theorems, notion of cusp forms
Sandeep Varma Professor TIFR Mumbai	3 lectures of 1 hours each	Introduction to the Spectral decomposition of $L^2(G/\Gamma)$ due to Langlands
Dipendra Prasad Professor IIT Bombay	1 lectures of 1 hours	Guest lecturer

Detailed Syllabus, and References:

(1) Prof. MS Raghunathan

Detailed Syllabus: The Peter Weyl theorem and its consequences for compact Lie groups, in particular the fact that a compact Lie group admits a faithful finite dimensional representation will be assumed. The first half of these lectures will be on the topology and structure of compact connected Lie groups. It will be shown that the Cohomology Algebra (over $\mathbb{R}$) of a compact Lie group G is an exterior algebra over a graded vector space with all graded components of odd degree. Using this it will be shown that for any integer $r \neq 0$, raising an element to its r^{th} power is a map of G onto itself and that all maximal tori in G are conjugates. Then it will be shown that a compact connected Lie group G is an almost direct product of a (central) torus and its commutator subgroup $[G, G]$. A compact connected Lie group is semisimple if it is equal to its commutator subgroup and $[G, G]$ is semisimple for any compact connected Lie group G. The second half of the lectures will be devoted to representations of a compact semisimple Lie group. The structure of the Lie algebra $\mathfrak{g}$ as well as its complexification $\mathfrak{g}^{\mathcal{C}_m}$ will be elucidated using the root-space decomposition of $\mathfrak{g}^{\mathcal{C}_m}$ with respect to $\mathfrak{t}^{\mathcal{C}_m}$, where $\mathfrak{t}$ is the Lie algebra of a maximal torus T and $\mathfrak{t}^{\mathcal{C}_m}$ is its complexification. Then there will be introduction of a lexicographic order on the dual of the real vector space $\mathfrak{h} = i \cdot \mathfrak{t}$ and description of the classification of irreducible representations in terms of the highest weight (in the above lexicographic order). The lectures will end with Weyl's formula for the character of an irreducible representation corresponding to the highest weight.

References:

1. C Chevalley, Theory of Lie Groups.
2. J F Adams, Compact Lie Groups.
3. Seminaire Sophus Lie.
4. Hermann Weyl, Math Zeitschrift 23(1925) 271-309, 24(1926) 328-376, 377-395, 787-791.
5. S Helgason, Differential Geometry, Lie Groups and Symmetric Spaces

(2) Prof. EK Narayanan

Detailed Syllabus: Basics of $SL(2, \mathbb{R})$, Infinite dimensional unitary representations, construction of prinicpal series, discrete series and complementary series, representations of Lie algebras, C^{∞} and analytic vectors, infinitesimal method, classification of irreducible representations of $SL(2, \mathbb{R})$, spherical functions, spherical inversion and Plancherel theorem.

References:

1) V. Bargmann: Irreducible unitary representations of the Lorentz group, Ann. of Math., 2 (48) 1947 568-640.
2) Serge Lang: $SL(2, R)$.
3) A. W Knapp : Representation theory of semisimple Lie groups. An overview based on examples.
4) Harish-Chandra: Plancherel formula for the 2 x 2 real unimodular group. Proc. Nat. Acad.Sci., 38, 1952, 337-342.

(3) Prof. Dragan Milicic

Detailed Syllabus: These lectures will be an introduction to study of the category of Harish-Chandra modules. First, there will be discussion on Harish-Chandra's approach to study of irreducible unitary representations of semisimple Lie groups. If G is a connected semisimple Lie group and K its maximal compact subgroup, Harish-Chandra proved that the restriction of an irreducible unitary representation of G to K is a direct sum of irreducible representations of K, and each such representation appears with finite multiplicity. This lead him to the definition of an admissible representation of G on a Banach space. The subspace of K-finite vectors of such representation consists of analytic vectors, hence it is a representation of the enveloping algebra of the complexified Lie algebra of G and also a representation of K. This leads to the notion of Harish-Chandra module.

Harish-Chandra modules are algebraic objects, and they constitute an abelian category. This category has a natural duality operation. To a Harish-Chandra module we can associate a unique matrix coefficient map which maps the tensor product of the Harish-Chandra module with its dual into real analytic functions on the group G. (In the case of Harish-Chandra module corresponding to a group representation, matrix coefficient map corresponds to the usual notion of matrix coefficients of a group representation).

Using matrix coefficient map one defines the character of a Harish-Chandra module which is a distribution on the group G. The character map factors through the Grothendieck group of the category of Harish-Chandra modules. Harish-Chandra studied power series expansion of the matrix coefficients around infinity in the group in two long unpublished manuscripts (which were later published in his collected works). These results were reinterpreted later in the paper of Casselman and Milicic where we show that they follow from some results of Deligne on differential equations with regular singularities.

Matrix coefficients contain information about analytic properties of a Harish-Chandra module. On the other hand if $G = KAN$ is the Iwasawa decomposition of the group G, Harish-Chandra expansions are closely related to the action of the complexified Lie algebra on N on the Harish-Chandra module. This observation leads to the proof of the Casselman subrepresentation theorem which states that an irreducible Harish-Chandra module is a submodule of a minimal principal series module (this is a strengthening of Harish-Chandra subquotient theorem). A refinement of this argument leads to Langlands classification of irreducible Harish-Chandra modules. There exist natural "completion" functors from Harish-Chandra modules into group representations due to Casselman, Wallach, Kashiwara and Schmid. Time permitting, some of their properties will be discussed.

References:

1) J.E. Humphries, Introduction to Lie Algebras and Representation Theory, Springer.
2) S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Mir. Math. Soc.

(4) Prof. R. Parthasarathy

Detailed Syllabus: Let G be a Real semisimple Lie group, and K be a maximal compact subgroup. The case of interest for discrete series is when G has a compact Cartan subgroup in K. Then the topics of discussion are: Invariant eigen distributions on G.

Schwartz space of G, Tempered invariant eigen distributions, Transition from the local theory on the Lie algebra $Lie(G)$ of G to the global theory on G, Construction of tempered invariant eigen distributions on G that will eventually turn out to be the distribution characters of irreducible discrete series representations of G(irreducible unitary representations of G by unitary operators on on infinite dimensional Hilbert space), Construction of the Tempered Invariant eigen distributions Θ_λ corresponding to a regular character λ on a compact Cartan subgroup.

To start with the distribution Θ_λ, it is easy to specify on the open set of regular elliptic elements resembling the Weyl character formula. The "Temperedness" condition and the differential equations need to be satisfied (ensuring an invariant eigen distribution) in set-up compelling circumstances imposing unique extension of the distribution from the regular elliptic set to the open set of all regular semisimple elements. It should be emphasized that the condition "distribution on G should satisfy the differential equations" is far more severe than "the same on G', the open set of all regular elements". The former imposes crucial compatibility conditions when one examines how the distribution extends along singularities between two disjoint connected Adjoint invariant open set of regular semisimple elements, and eventually leads to the uniqueness of extension from the elliptic regular set. There will be attempt to sketch some of these features. It may be too ambitious to launch upon how the K-finite Fourier coefficients of Θ_λ suffice to span the L^2-closure of the matrix coefficients of discrete series representations of G. Time permitting an attempt may be made to indicate the first few essential steps - Orbital Integrals and Cusp Forms, if it does not overlap with the topics of other speakers.

References:
The 3 talks will be based mostly on a longer series of expository talks of V.S. Varadarajan "The Theory of Characters And The Discrete Series For Semisimple Lie Groups" (1973 AMS) on these topics. V.S.V 's exposition is a true portrayal of Harish-Chandra's early ground work in three fundamental papers in Transactions of the AMS in the early 50's and his seminal papers "Discrete Series I" and "Discrete Series II".

(5) Prof. Arvind Nair

Detailed Syllabus: Constructing and classifying all unitary representations of real reductive groups with cohomology, in particular introducing the representations $A_q(\lambda)$ through the process of cohomological induction due to Zuckerman. The attempt would be to give an idea of how the theory works, without too much by way of technical detail mainly from the 2nd reference. Some of the prerequisites would be Structure theory of non-compact Lie groups. Some basic familiarity with (g, K)-modules, Some familiarity with the passage from (unitary) representations to (unitarizable) (g, K)-modules would be helpful in understanding the motivation for some constructions, but it can be taken as a black box, and Basic homological algebra (derived functors, injective resolutions etc.).

References:
(1) Vogan and Zuckerman, Unitary representations with non-zero cohomology, Compositio 53 (1984).
(2) Knapp and Vogan, Cohomological Induction and Unitary Representations, Princeton

Univ. Press (1995).

(6) Prof. Sandeep Varma

Detailed Syllabus: A crude idea of the following topics will be given: Cuspidal representations., Eisenstein series, An informal description of the spectral decomposition of the right-regular action of $G(A)$ on $L^2(G(\mathbb{Q})\backslash G(A))$, where G is a reductive group over $\mathbb{Q}$ and A denotes the ring of adeles over $\mathbb{Q}$.

Familiarity with Fourier series and with Fourier transform on $\mathbb{R}$, basic forms of spectral theory such as the description of compact self-adjoint operators, a basic knowledge of semisimple Lie groups or reductive algebraic groups, and Lie algebras, basic familiarity with adeles and ideles will be required.

References:
J. Arthur, Eisenstein series and the trace formula, in: Automorphic Forms, Representations and L-functions, Part 1, Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, OR, 1977, in: Proc. Sympos. Pure Math., vol. XXXIII, Amer. Math. Soc., Providence, RI, 1979, pp. 253–274

(6) Prof. Ravi Raghunathan

Detailed Syllabus: After a brief review of reduction theory, the spaces of automorphic forms and cusp forms will be defined (with a focus on $GL(n)$). The main aims of the three lectures will be to prove a theorem of Harish-Chandra that the space of automorphic forms of a given type is finite dimensional and a theorem of Gelfand and Piatetski-Shapiro decomposing the space $L^2_0(G/\Gamma)$ (the space of cusp forms) as a Hilbert direct sum of representations occurring with finite multiplicities.

References:
(1) Automorphic forms on reductive groups. Automorphic forms and applications, 7–39, IAS/Park City Math. Ser., 12, Amer. Math. Soc., Providence, RI, 2007.
(2) Bump, Daniel Automorphic forms and representations. Cambridge Studies in Advanced Mathematics, 55. Cambridge University Press, Cambridge, 1997. xiv+574 pp. ISBN: 0-521-55098-X.
(3) Automorphic representations and L-functions for GL(n). The genesis of the Langlands Program, 215–274, London Math. Soc. Lecture Note Ser., 467, Cambridge Univ. Press, Cambridge, 2021.

Conveners :

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