Mathematics Ph.D. Course Work Syllabus

The course is of total of 60 credits, as described below, and will spread over two semesters.

Course structure: Algebra-I, Analysis-I, Topology-I and the Differentiable Manifolds courses will be taught in the first semester. In the second semester Algebra-II, Analysis-II and Topology-II will be taught. It was decided to put the differentiable manifolds in the first semester, based on the input from students and the fact that the topics in the second semester are more advanced than the first semester. In addition to these, there will be one project course and one RM course in the first year itself.

Credits: Each of the above courses will have minimum 45 hours of lectures and 30 hours of tutorials (2 hours of tutorial per week). Hence each of these seven courses will be of 8 credits. Together with a project course for 4 credits, the course work will be of total 60 credits. Each of the courses will be of 100 marks, except the project course which will have only 50% weightage, being a 4 credit course.

Project: The student will be encouraged to take a project course in the first semester itself, so that if uncomfortable in the area, s/he will have an option to try out some other area in the second semester. This will save time for the students, to fix a thesis supervisor soon after their OGC Exam is over.

Course Levels: All courses including core courses, electives and project are of level 700 and above.

The proposed syllabi for the courses is listed below, including the elective courses.

Algebra I (Level 700, Total contact hours : 30 x 1.5 hours = 45 hours)

Group Theory [5 Lectures]
The Jordan-Hölder Theorem, Solvable groups and nilpotent groups, Symmetric and alternating groups, Groups acting on sets, Applications: The Sylow Theorems, the (conjugacy) class equation, Free groups

Category theory [5 Lectures]
Categories and functors, Natural transformations, Equivalence of categories and Adjoint functors, Universal objects, Representable functors, Yoneda lemma, Free product of groups - Product and co-product, Limits and colimits

Rings and Modules [10 Lectures]
Ideal and quotient rings, Integral domain, principal ideal domains (PID) and unique factorisation domains (UFD), Fraction ideal and Dedekind domains, Polynomial ring and polynomial ring over a factorial ring, Symmetric polynomials, Noetherian and Artinian conditions, The Hilbert Basis Theorem, Direct sum and direct product of modules, Bilinear maps and forms, The tensor product, Inductive and projective limits of rings and modules

Field Theory [10 Lectures]
Finite and algebraic extensions, The Steinitz Theorem on algebraic closures, Splitting fields, Normal and separable extensions, Primitive element theorem, Finite fields, Galois extensions and Galois groups, Main theorem of Galois theory, Examples: Quadratic, Cubic and Cyclotomic extensions, Cyclic extensions, Solvable and radical extensions

Textbooks:
1. S. Lang, Algebra, Revised third edition.
2. M. Artin, Algebra.
3. D.S. Dummit and R. M. Foote, Abstract Algebra, Second edition.
4. P.M. Cohn, Basic Algebra.

5. T.W. Hungerford, Algebra.
6. M.P. Murthy et. al., Galois Theory, TIFR Pamphlet No. 3.
7. N. Jacobson, Basic Algebra, Vols. 1 and 2.
8. S. Bosch, Algebra (from the viewpoint of Galois Theory).

Algebra II (Level 700, Total contact hours : 30 x 1.5 hours = 45 hours)

Multilinear and homological algebra [12 Lectures]
Review of Module theory: Direct sum & product, modules generated by a set, Commutative diagram, snake lemma, five lemma, Finiteness conditions: finite generation, finite presentation, Noetherian, Coherent, Artinian modules, Limits and colimits of modules, Review of tensor products, tensor-Hom adjunction, Right exactness of tensoring, Flat and faithfully flat modules, Left exactness of Hom, Injective and projective objects, Torsion-free, divisibility and injectivity, Baer's criterion, Relation between finite presentation and projectivity, flatness, Algebras, Tensor, symmetric and exterior algebras

Commutative algebra [12 Lectures]
Rings and modules of fractions, Local rings, Nakayama's lemma, Integral extensions, Transcendence degree, Noether's normalisation theorem, Hilbert's Nullstallensatz, Discrete valuations rings, Dedekind domains, Primary decomposition, (Yoneda) Ext and Tor of modules

Linear algebra [6 Lectures]
Finitely generated Modules over principal ideal domains (if possible over Dedekind domains), The minimal polynomial of an endomorphism, The Jordan canonical form, The characteristic polynomial of an endomorphism, The Cayley-Hamilton Theorem

Textbooks:
1. S. Lang, Algebra, Revised third edition.
2. M.F. Atiyah and I.G. McDonald, Introduction to Commutative Algebra.
3. P.M. Cohn, Basic Algebra.
4. N. Bourbaki, Algebra, Chapters 2, 3 and 7.
5. N. Bourbaki, Commutative Algebra, Chapters 1 to 6.
6. N. Jacobson, Basic Algebra, Vol. 2.
7. S. Bosch, Algebraic Geometry and Commutative algebra, Part A.

Analysis I (Level 700,, Total contact hours : 30 x 1.5 hours = 45 hours)

Measure Theory [17 Lectures]
Basic properties of measures, Outer measures, Caratheodory's theorem, completeness of measures, Borel measures on R, Measurable functions, Approximations by simple functions, in the sense of a.e. and L^1, integration of real valued functions, Fatou's lemma, integration of complex functions, Convergence theorems.Various modes of convergence - convergence in measure, a.e. and L^1 convergence, and the implications, Product sigma algebras, Fubini-Tonelli's Thorem, N-dimensional Lebesgue integral, Properties of Lebesgue measure-translation invariance, change of variable formula for Lebesgue integral , polar co-ordinates etc., Signed measures, Hahn decomposition theorem, Jordan Decomposition theorem, Lebesgue Radon Nikodym Theporem, Complex measures, Fundamental Theorem of calculus for Lebesge integrals

L^p spaces and Fourier transform [5 Lectures]
L^p spaces, completeness, Dual spaces, converse of Holder's inequality. Convolutions of functions, Fourier transform, Riemann Lebesgue lemma, Fourier inversion in Schwartz space, Interaction of Fourier transform and convolutions, The stone Weiestrass Theorem

Calculus on normed linear spaces [8 Lectures]
Normed vector spaces and Banach spaces, Bounded linear maps between Banach spaces, Dual of a Banach space, Riesz Lemma, Continuous multi-linear maps, Differentiable mappings, continuous differentiability, Higher derivatives.Differentiation on finite dimensional spaces, and Partial differentiation, Chain rule of differentiation, Mean value theorem and applications, Inverse function theorem, Implicit function theorem

Reference books.
1. G. B. Folland, Real Analysis; Modern techniques and their applications, Second edition.
2. W. Rudin, Real and Complex Analysis, Third edition.
3. H. Cartan, Differential calculus.
4. S. Lang, Real analysis, Second edition.
5. E. Stein and R. Shakarchi, Real Analysis; Measure theory, integration, and Hilbert spaces.

Analysis II (Level 700, Total contact hours : 30 x 1.5 hours = 45 hours)

Elementary Functional analysis [6 Lectures]
Arzela-Ascoli theorem, Hahn Banach Theorems, Open mapping and closed graph theorem, Uniform boundedness principle. Hilbert spaces, existence of orthonormal basis, Riesz theorem for linear functionals

Topological Vector spaces [10 Lectures].
Topological Vector spaces (TVS), Local bases, Types of TVS and basic properties, Finite dimensional vector spaces, Metrizability, Boundedness and continuity of linear maps, Minkowski functionals and Semi norms, Locally convex spaces, Quotient spaces, Examples of TVS, Bi-linear Maps, Dual space of a TVS, Continuity of linear functionals, Weak topologies

Banach Algebras and Special Theory [14 Lectures]
Vector valued Holomorphic functions, Banach algebras, Complex Homomorphisms, Multiplicative linear functionals, Spectrum of a Banach algebra, Gelfand-Mazur Theorem, Symbolic calculus, Integration of Banach algebra valued functions, Holomorphic Banach algebra valued functions and Cauchy's theorem, Banach algebra of bounded operators on a Banach space X, Group of invertible elements. Commutative Banach algebras, Gelfand transform, Gelfand -Naimark Theorem, Compact Operators, Fredholm Theory

Text books :
1. W. Rudin, Functional Analysis, Second edition.
2. K. Yosida , Functional Analysis, Reprint of the sixth (1980) edition.
3. F. Treves, Topological Vector Spaces, Distributions and Kernels.
4. E. Stein and R. Shakarchi, Functional Analysis; Introduction to further topics in analysis.
5. V.S. Sunder, Functional Analysis; Spectral Theory.

Topology-1 (Level 700, Total contact hours : 30 x 1.5 hours = 45 hours)

Topological spaces [7 Lectures]
Topologies; bases; continuous maps; subspaces; quotient spaces; products; connectedness and compactness; proper maps, *Convergence*: the relations between convergence and countability and separation axioms; relations with compactness and proper maps.

Topological groups and Metrisability [7 Lectures]

Topological groups; uniform structures; products of compact spaces; actions; orbit spaces; proper actions; homogeneous spaces. Metrisability and paracompactness; complete metric spaces; function spaces. Inductive and projective limits: Inductive and projective limits of topological spaces

Homotopy theory [12 Lectures]

Homotopy; retraction and deformation; suspension; mapping cylinder; fundamental group; the Van Kampen Theorem; etale spaces; covering spaces; homotopy lifting property; relations with the fundamental group; lifting of maps; universal coverings; automorphisms of a covering; Galois coverings

Simplicial topology [4 Lectures]

Simplicial complexes; triangulations

Textbooks:
1. N. Bourbaki, General topology, Vol. 1.
2. E. Spanier, Algebraic topology (1995).
3. W. S. Massey, Algebraic topology : An introduction (1977).
4. A. Hatcher, Algebraic Topology.
5. W. Fulton, Algebraic Topology, A first course.
6. J. Munkres, Elements of Algebraic Topology.

Topology II (Level 700, Total contact hours : 30 x 1.5 hours = 45 hours)

Homology [18 Lectures]

Simplicial homology; singular homology; the Mayer-Vietoris sequence; The Jordan-Brouwer Separation Theorem; the Universal Coefficient Theorem; the Kunneth Formula; CW complexes; cellular homology and computations for projective spaces; the Lefschetz Fixed Point Theorem.

Cohomology [12 Lectures]

Singular cohomology; the Universal Coefficient Theorem; the Kunneth Formula; cup and cap products; Poincaré duality for a topological manifold.

Textbooks:
1. E. Spanier, Algebraic topology (1995).
2. M. J. Greenberg and J. R. Harper, Algebraic topology : A first course.
3. A. Hatcher, Algebraic Topology.
4. G. E. Bredon, Topology and Geometry (1997).
5. J. W. Vick, Homology Theory : An Introduction to Algebraic Topology, Second edition.

Differentiable Manifolds (Level 700, Total contact Hours : 30 x 1.5 hours = 45 hours)

Differentiable manifolds [10 Lectures]

Basic notions; the effects of second countability and Hausdorffness; smooth maps; tangent and cotangent spaces; sub manifolds; consequences of the Inverse Function Theorem; vector fields and their flows; the Frobenius Theorem; Sard’s theorem.

Differential forms [8 Lectures]

Recapitulation of multilinear algebra; tensors; differential forms; the de Rham complex and its behaviour under differentiable maps; the Lie derivative; differential ideals.

Integration on manifolds [8 Lectures]
Orientation; the integral of differential forms on differentiable singular chains; integration of differential forms of top degree on an oriented differentiable manifold; the theorems of Stokes; the volume form on an oriented Riemannian manifold; the divergence theorem; integration on a Lie group.

de Rham cohomology [4 Lectures]
Definition; real differentiable singular cohomology; statement of the de Rham theorem; the Poincaré lemma.

Textbooks:
1. F. W. Warner, Foundations of differentiable manifolds and Lie groups.
2. I. Madsen and J. Tornehave, From calculus to cohomology.
3. John M. Lee, Introduction to smooth manifolds; de Rham cohomology and characteristic classes.

<u>**Elective courses** (Each has level 700 or above):</u>

A student can take up a research project under any one of the following broad areas. This may involve lectures in the second year, or a reading course with discussion, as decided by the thesis advisor.

Algebraic number theory : p-adic numbers, p-adic valuations, absolute values, completions, local fields, henselian fields, extensions of valuations, ramification, higher ramification groups. Galois extensions, projective and inductive limits, abstract class field theory, the Herbrand quotient. The local reciprocity law, the norm residue symbol, formal groups, cyclotomic extensions, compatibility with the ramification filtration. Ideles and idele classes, ideles in extensions, global class field theory, power reciprocity laws.

Textbooks:
1. A. Fröhlich and M. J. Taylor, Algebraic number theory.
2. H. Hasse, Number theory, Translated from the third (1969) German edition, Reprint of the 1980 English edition.
3. J. Neukirch, Algebraic number theory, Translated from the 1992 German original.
4. A. Weil, Basic number theory, Reprint of the second (1973) edition.

Analytic number theory: Arithmetical functions, Euler, Abel Summation formula and applications to summatory functions. The Riemann Zeta function, and Dirichlet L-functionsThe Prime Number Theorem, and Dirichlet's Prime Number Theorem Sieve methods (Brun, Selberg, and Large), and their applications Fourier techniques and applications in number theory.

Textbooks:
1. K. Chandrasekharan, Introduction to analytic number theory.
2. H. Iwaniec, and E. Kowalski, Analytic number theory.

Algebraic varieties: Spaces with sheaves of functions, affine algebraic varieties over an algebraically closed field, the category of algebraic varieties, subvarieties, products, projective varieties, separation, normality, dimension, rational maps, tangent spaces, smoothness, completeness, finite morphisms, constructible sets, divisors, curves, and the Riemann-Roch theorem.

Textbooks:
1. G. R. Kempf, Algebraic varieties.
2. D. Eisenbud, Commutative algebra; With a view toward algebraic geometry.
3. J. S. Milne, Algebraic geometry, http://www.jmilne.org/math/.

Commutative algebra: Modules and tensor products, prime ideals, the Zariski topology, rings and modules of fractions, flatness, valuation theory, integral extensions, discrete valuation rings, Dedekind domains, Artinian and Noetherian rings and modules, the Hilbert basis theorem, primary decomposition, Noether normalization, Hilbert's Nullstellensatz, completions, the Krull dimension

Textbooks:
1. S. Bosch, Algebraic geometry and commutative algebra.
2. D. Eisenbud, Commutative algebra; With a view toward algebraic geometry.

Introduction to Elliptic curves: Elliptic curve and rational points, group law, elliptic curves over a finite field, elliptic curves over complex numbers, elliptic curves over local and global fields, Mordel-Weil theorem, Selmer group, and Tate-Shafarevich group.

Textbook:
1. J. H. Silverman, The Arithmetic of Elliptic Curves.

p-adic Numbers, p-adic Analysis, and Zeta-Functions: p-adic numbers, p-adic interpolation of Riemann's zeta function, p-adic power series, rationality.

Textbook:
1. N. Koblitz, p-adic Numbers, p-adic Analysis, and Zeta-Functions, Second edition.

Introduction to Iwasawa theory: Cyclotomic fields, local units, Iwasawa algebras, and p-adic measures, cyclotomic unites, Euler system, Main conjecture.

Textbooks:
1. J. Coates and R. Sujatha, Cyclotomic Fields and Zeta Values.
2. L. C. Washington, Introduction to Cyclotomic Fields.

Function Field Arithmetic: Number fields and Function fields, Drinfeld modules, Explicit class field theory of Drinfeld modules, Gamma functions, Zeta functions.

Textbook:
1. D. S. Thakur, Function Field Arithmetic.

Introduction to p-adic Galois representations: Absolute Galois groups of non-archimedean local fields, classification of p-adic Galois representations in terms of certain objects from semilinear algebra, the so-called étale \varphi - and (\varphi, \Gamma)-modules.

Textbooks:
1. J.-M. Fontaine and Y. Ouyang, Theory of p-adic Galois representations, preprint.
2. L. Berger, An introduction to the theory of p-adic representations.

Local Fields: Discrete valuation rings and Dedekind domains, completions, discriminant and different, ramification groups, the norm, Artin representation, group cohomology, Galois cohomology, class formations, Brauer groups, local class field theory.

Textbooks:
1. J. W. S. Cassels, Local fields.
2. I. B. Fesenko and S. V. Vostokov, Local fields and their extensions, Second edition.
3. K. Iwasawa, Local class field theory.
4. J.-P. Serre, Local fields. Translated from the French original.

Fourier analysis: Lp spaces, basic inequalities including Holder's, Chebyshev, and Minkowski, inequalities for integrals, weak Lp spaces, the Riesz-Thorin and Marcinkiewicz Interpolation Theorems Convolution, Young's inequality, the generalized inequality of Young, approximations to the identity, the Fourier transform, Hausdorf-Young inequality, the Riemann Lebesgue Lemma the Fourier Inversion Theorem, the Plancherel Theorem, summability theorems, including those of Cesaro and Fejer, Fourier inversion on the torus, the Riemann localisation principle, Distributions, Sobolev spaces, Sobolev Embedding Theorem, Rellich's theorem, applications to basic linear PDE

Textbooks:
1. G. B. Folland, Real analysis; Modern techniques and their applications, Second edition.
2. W. Rudin, Functional analysis, Second edition.

Harmonic analysis: Maximal Function, the Riesz-Thorin Interpolation Theorem. Singular integrals, the Calderon-Zygmund Decomposition, Singular integral operators which commute with dilations, vector-valued analogues, Riesz transforms, Poisson integrals, approximate identities, spherical harmonics The Little wood-Paley g-function, multipliers, dyadic decomposition, the Hormander- Mihilin Multiplier Theorem, the Marcinkiewicz Multiplier Theorem.

Textbooks:
1. J. Duoandikoetxea, Fourier analysis, Translated and revised from the 1995 Spanish original.
2. E. M. Stein, Singular integrals and differentiability properties of functions.

Advanced Complex Analysis : Basic properties of holomorphic functions; relations with the fundamental group and covering spaces; the open mapping theorem; the maximum modulus theorem; zeros of holomorphic functions; classification of singularities; meromorphic functions; the Weierstrass factorization theorem; Riemann mapping theorem; the Little Picard theorem. Montel's theorem, Cauchy's integral formula, homotopy form of Cauchy's theorem, argument principle and Hurwitz's theorem. Jensen's formula, entire functions of finite order, Weierstrass infinite products, Hadamard's factorisation theorem, Phrageman-Lindelof theorem Runge's approximation theorem, Mittag Leffler theorem, cohomology form of Cauchy's theorem, Ahlfors version of Schwarz Lemma, Big Picard's theorem, Basic properties of holomorphic/ meromorphic functions on Riemann surface (principle of analytic continuation, open mapping theorem, maximum principle, Weierstrass theorem, Montel's theorem, Riemann extension theorem), Compact Riemann surface associated to irreducible algebraic polynomial in two variables, Finitess of first cohomology group associated to compact Riemann surface and existence of non-constant meromorphic function on a Compact Riemann surface. Riemann-Roch theorem and applications.

Textbook:
1. R. Narasimhan and Y. Nievergelt, Complex Analysis in one variable, Second edition.

Introduction to number theory: Divisibility, congruence, the Fundamental Theorem of Arithmetic, the Chinese Remainder Theorem. Elementary proofs of the infinitude of primes in certain arithmetic progressions, the quadratic reciprocity law. The Bertrand postulate, the Euler and Abel summation formulas and applications, preparation for the Prime Number Theorem Combinatorial sieves including the Brun sieve and the Turan sieve, and their applications. Rational approximations, the Dirichlet Theorem, the Liouviile Theorem, Siegel's Lemma, and their applications, Gaussian integers, sums of squares.

Textbooks:
1. D. M. Burton, Elementary number theory, Second edition.
2. K. Chandrasekharan, Introduction to analytic number theory.
3. A. C. Cojocaru and M. R. Murty, An introduction to sieve methods and their applications.
4.. G. H. Hardy and E. M. Wright, An introduction to the theory of numbers, Sixth revised edition.

Lie algebras: Linear Lie algebras, Lie algebras of derivations, homomorphisms of Lie algebras, representations of Lie algebras, solvable and nilpotent Lie algebras, Engel's Theorem, Lie's Theorem, the Jordan-Chevalley decomposition. Cartan's criterion, the Killing form, criterion for semi-simplicity, inner derivations, the abstract Jordan decomposition, modules, the Casimir element, Weyl's Theorem, preservation of Jordan decomposition, Representations of sl(2, F), maximal toral subalgebras, and roots Orthogonality, integrality and rationality properties of roots, root systems, bases, Weyl chambers, the Weyl group, irreducible root systems, the Cartan matrix. Coxeter graphs, Dynkin diagrams, the Classification Theorem, construction of root systems & automorphisms, the theory of weights, dominant weights, the Isomorphism Theorem, Cartan subalgebras, the universal enveloping algebra, the Poincare-Birkhoff- Witt Theorem.

Textbooks:
1. R. W. Carter, Lie algebras of finite and affine type.
2. J. E. Humphreys, Introduction to Lie algebras and representation theory, Second printing, revised.
3.A. W. Knapp, Lie groups beyond an introduction, Second edition.

Representations of finite groups: Representations, irreducible and indecomposable representations, class functions, orthogonality relations for characters, character tables, Schur's lemma, unitary representations, duals and tensor products of representations, regular representations, canonical decompositions, examples, induced representations, Mackey's criterion, Frobenius reciprocity, group algebras, Maschke's theorem, applications of the representation theory of finite groups. Artin's theorem, Brauer's theorem and applications.

Textbooks:
1. W. Fulton and J. Harris, Representation theory : A first course.
2. J.-P. Serre, Linear representations of finite groups, Translated from the second French edition.

Other elective courses: Apart from the list above, there may be elective courses in special Topics in Algebra/Analysis/Topology/ Algebraic Geometry/Differential Geometry based on the interest of students. If a faculty member opts to give such a course, the syllabus and text books will be announced well in advance.