RESEARCH STATEMENT

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I briefly mention my research work done upto December 2018. Some of the sections are divided into several subsections according to different publications. Each section contains results from the articles sharing the theme of the work. We have recently published a monograph [18] (jointly with M. Singh and IBS Passi) and edited a volume [21] ((jointly with NSN Sastry). The monograph is about automorphisms of finite groups which reveals a connection between the order of a Sylow p-subgroup of a group G and the order of a Sylow p-subgroup of the automorphism group of G, and the volume discusses recent trends in group theory and computational methods. I have placed the papers involving me as an author at the beginning of the bibliography. Rest of the bibliography is given in alphabetical order.

1. Class-Preserving Automorphisms

Let G be a finite group. An automorphism f of G is called (conjugacy) class-preserving if $f(g) \in g^G$ for all $g \in G$, where g^G denotes the conjugacy class of g in G, i.e., the set of all conjugates $x^{-1}gx$ of g in G. Let $\operatorname{Aut}(G)$ denote the group of all automorphisms of G. Then the set of all class-preserving automorphisms of G is a normal subgroup of $\operatorname{Aut}(G)$. We denote this subgroup by $\operatorname{Aut}_c(G)$. An automorphism ι is called an inner automorphism if there is an element $x \in G$ such that $\iota(g) = x^{-1}gx$ for all $g \in G$. An automorphism of G which is not an inner automorphism is called an outer automorphism of G. The set of all inner automorphisms of G, denoted by $\operatorname{Inn}(G)$ is a normal subgroup of $\operatorname{Aut}(G)$ such that $\operatorname{Inn}(G) \subseteq \operatorname{Aut}_c(G)$. Set $\operatorname{Out}_c(G) = \operatorname{Aut}_c(G)/\operatorname{Inn}(G)$. We call this factor group $\operatorname{Out}_c(G)$, the group of all class-preserving outer automorphisms of G. Clearly $\operatorname{Out}_c(G)$ is a normal subgroup of $\operatorname{Out}(G) = \operatorname{Aut}(G)/\operatorname{Inn}(G)$ - the group of all outer automorphisms of G.

1.1. Groups with maximal $\operatorname{Aut}_c(G)$.

In this subsection we mentions results from the following two papers:

- *Class-preserving automorphisms of finite p-groups*, J. Lond. Math. Soc. (2) **75** (2007), no. 3, 755-772.
- *Class-preserving automorphisms of finite p-groups II*, Israel J. Math. **209** (2015), 355-396

In 1911, W. Burnside [46, pg. 463] posed the following question: Does there exist any finite group G such that G has a non-inner class-preserving automorphism? In 1913, Burnside [47] himself gave an affirmative answer to this question. He constructed a group G of order p^6 isomorphic to the group W consisting of all 3×3 matrices

$$M = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}$$

with x, y, z in the field $\mathbb{F}_{p^2}$ of p^2 elements. For this group G, $\operatorname{Inn}(G) < \operatorname{Aut}_c(G)$. He also proved that $\operatorname{Aut}_c(G)$ is an elementary abelian p-group of order p^8. This group G is represented by the group W.

We prove the following theorem, which provides a very neat bound for $|\operatorname{Aut}_c(G)|$:

Theorem 1.1. *Let G be a non-trivial p-group having order p^n. Then*

$$|\operatorname{Aut}_c(G)| \leq \begin{cases} p^{\frac{(n^2-4)}{4}}, & \text{if } n \text{ is even;} \\ p^{\frac{(n^2-1)}{4}}, & \text{if } n \text{ is odd.} \end{cases} \tag{1.2}$$

One can easily notice that equality holds in (1.2) for the group W. Motivated from this we have the following natural problem:

Problem. Classify all finite p-groups G such that equality holds in (1.2).

We use the term 'isoclinic' in the sence of P. Hall [69]. Two finite groups G and H are called *isoclinic* if there exist an isomorphism i of the factor group $\bar{G} = G/\operatorname{Z}(G)$ onto $\bar{H} = H/\operatorname{Z}(G)$, and an isomorphism j of the subgroup $[G,G]$ onto $[H,H]$, such that i and j carry the map $c_G\colon \bar{G} \times \bar{G} \to [G,G]$ to the map $c_H\colon \bar{H} \times \bar{H} \to [H,H]$, in the sense that

$$c_H\big(i(\bar{x}), i(\bar{y})\big) = j\big(c_G(\bar{x}, \bar{y})\big) \in [H,H] \tag{1.3}$$

for all $\bar{x}, \bar{y} \in \bar{G}$, where $c_G(\bar{x}, \bar{y}) = [x, y]$ and c_H is defined similarly. The resulting pair (i, j) is called an *isoclinism* of G onto H. A group G is called a *Camina group* if each non-trivial coset $x\gamma_2(G)$ of the derived group $\gamma_2(G)$ is a single conjugacy class x^G in G [48]. We solve the above problem in the following theorem:

Theorem 1.4. *Let G be a non-abelian finite p-group of order p^n. Then equality holds in* (1.2) *if and only if one of the following holds:*

(1.5a) *G is an extra-special p-group of order p^3;*

(1.5b) *G is a group of nilpotency class 3 and order p^4;*

(1.5c) *G is a Camina special p-group isoclinic to the group W and $|G| = p^6$;*

(1.5d) *G is isoclinic to R and $|G| = p^6$,*

where R is the group $\phi_{21}(1^6)$ in the isoclinism family (21) *of* [83]*:*

$$\begin{aligned} R \;\; = \;\; & \langle \alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2 | [\alpha_1, \alpha_2] = \beta, [\beta, \alpha_i] = \beta_i, [\alpha, \alpha_1] = \beta_2, \\ & [\alpha, \alpha_2] = \beta_1^{\nu}, \alpha^p = \beta^p = \beta_i^p = 1, \alpha_1^p = \beta_1^{\binom{p}{3}}, \alpha_2^p = \beta_1^{-\binom{p}{3}}, \\ & i = 1, 2 \rangle, \end{aligned}$$

where ν is the smallest positive integer which is a non-quadratic residue $\mod p$ *and β_1 and β_2 are central elements.*

More generally, for a finite group G minimally generated by d elements $x_1, x_2, \ldots, x_d$, it follows that

$$|\operatorname{Aut}_c(G)| \le \prod_{i=1}^{d} |x_i^G|, \tag{1.6}$$

since there are no more choices for the generators to go under any class-preserving automorphism. Notice that (1.6) holds true for any minimal generating set $\{x_1, x_2, \ldots, x_d\}$ for G. Since $|x^G| = |[x, G]| \le |\gamma_2(G)|$, from (1.6) we get

$$|\operatorname{Aut}_c(G)| \le |\gamma_2(G)|^d. \tag{1.7}$$

Let us get back to the group W (defined above). Notice that $|\gamma_2(W)| = p^2$ and $d(W) = 4$, and W enjoys the following two properties: (i) W is a special p-group, i.e., $\gamma_2(W) = \mathrm{Z}(W) = \Phi(W)$; (ii) $[x, W] = \gamma_2(W)$ for all $x \in W - \gamma_2(W)$. Thus it is not difficult to see that for the group W equality holds in (1.6) as well as in (1.7). Are there other groups G for which such a property on class-preserving automorphisms holds true? The answer is yes. First of all we notice that the field $\mathbb{F}_{p^2}$ of p^2 elements, used in the definition of W, is nothing special with respect to the properties (i), (ii) and the condition on the size of the class-preserving automorphism group. It follows from the first part of Subsection 1.2 that the group G consisting of all 3×3 unitriangular matrices over a finite field $\mathbb{F}_{p^m}$ of p^m elements, where $m \ge 2$ and p is an odd prime, satisfies properties (i) and (ii), and $|\operatorname{Aut}_c(G)| = |\gamma_2(G)|^{d(G)} = \prod_{i=1}^{d(G)} |x_i^G|$ for all minimal generating sets $\{x_1, \ldots, x_{d(G)}\}$ of G. Notice that G is a Camina groups and W is isomorphic to this group G for $m = 2$. It follows from the definition that equality holds for Camina p-group G of nilpotency class 2 in (1.6) and (1.7) simultaneously.

We asked ourself: Is there anything special about Camina p-groups of class 2 so that equality holds in (1.6) and (1.7) simultaneously? Our answer is: No, there is nothing special about Camina groups regarding these equalities to hold true simultaneously. It is, in fact, a general phenomenon in all finite p-groups. We show this interesting fact in the following theorem.

Theorem 1.8. *Let G be a finite p-group. Then equality holds in* (1.6) *for all minimal generating sets $\{x_1, \ldots, x_d\}$ of G if and only if equality holds for G in* (1.7)*.*

Having handful of examples of groups for which equality hold in (1.7), a natural question which arises here [26, Problem 6.7] is the following:

Question. What are all other groups for which equality hold in (1.7)?

A special case of this question is considered in Theorem 1.4 above. We now study this question in full generality for finite nilpotent groups. Since a finite nilpotent group can be written as a direct product of its Sylow p-subgroups for suitable prime numbers p, and the automorphism group of such groups is direct product of the automorphism groups of corresponding Sylow p-subgroups, it is sufficient to study this question for finite p-groups. For the sake of simplicity, instead of saying that we study finite p-groups for which equality hold in (1.7), we say that we study finite p-groups G which satisfy the following hypothesis:

Hypothesis A 1.9. *Equality holds for G in* (1.7) *(or equivalently, equality holds in* (1.6) *for all minimal generating sets $\{x_1, \ldots, x_d\}$ of G).*

However, a complete classification of finite p-groups of class 2 satisfying Hypothesis A seems a very difficult task (as shown by Camina groups), we obtain some structural information of these groups given in the following theorem.

Theorem 1.10. *Let G be a finite p-group of nilpotency class* 2 *satisfying Hypothesis A. Then the following hold true:*

(i) $[x, G] = \gamma_2(G)$ for all $x \in G - \Phi(G)$;

(ii) $d(G)$ is even and $d(G) \geq 2d(\gamma_2(G))$;

(iii) $G/\operatorname{Z}(G)$ is homocyclic.

On the other hand, if G is a finite p-group of class 2 *such that (i) holds true for G, then G satisfies Hypothesis A.*

Fortunately, the situation is much more satisfying and interesting for p-groups of nilpotency class larger than 2. By the way, we didn't mention yet any example of groups of nilpotency class larger than 2 which satisfies Hypothesis A. Let us first do this. Consider the metacylic group

$$K := \left\langle x, y \mid x^{p^{r+t}} = 1, y^{p^r} = x^{p^{r+s}}, [x, y] = x^{p^t} \right\rangle, \tag{1.11}$$

where $r \geq 2$, $1 \leq t \leq r$ and $0 \leq s \leq t$. Notice that for $r > t$, the nilpotency class of K is at least 3. Since K is generated by 2 elements, it follows from (1.7) that $|\operatorname{Aut}_c(K)| \leq |\gamma_2(K)|^2 = p^{2r}$. It is not so difficult to see that $|\operatorname{Inn}(K)| = |K/\operatorname{Z}(K)| = p^{2r}$. Since $\operatorname{Inn}(K) \leq \operatorname{Aut}_c(K)$, it follows that $|\operatorname{Aut}_c(K)| = |\gamma_2(K)|^2 = |\gamma_2(K)|^{d(K)}$. Thus K satisfies Hypothesis A. Furthermore, if H is any 2-generated group isoclinic to K, then it follows that H satisfies Hypothesis A. Are there any more example? The answer, surprizingly, is in negative for odd p. We prove this in the following theorem.

Theorem 1.12. *Let G be a finite p-group of nilpotency class at least* 3*. Then the following statements hold true:*

(i) *If G satisfies Hypothesis A, then $d(G) = 2$.*

(ii) *Let $|\gamma_2(G)/\gamma_3(G)| > 2$. Then G satisfies Hypothesis A if and only if G is a* 2*-generator group with cyclic commutator subgroup. Moreover, G is isoclinic to the group K defined in* (1.11) *for suitable parameters.*

(iii) *Let $|\gamma_2(G)/\gamma_3(G)| = 2$. Then G satisfies Hypothesis A if and only if G is a* 2*-generator* 2*-group of nilpotency class* 3 *with elementary abelian $\gamma_2(G)$ of order at most* 8.

1.2. **Unitriangular groups and p-groups of orders $\leq p^6$.** In this subsection we mention results from the following papers:

- *Class-preserving automorphisms of unitriangular groups*, Internat. J. Algebra Comput. **22**, (2012), 17 pages, (jointly with V. Bardakov and A. Vesnin).
- *"Hasse principle" for extraspecial p-groups*, Proc. Japan Acad. **76**, Ser. A (2000), 123-125 (jointly with L. R. Vermani).
- *"Hasse principle" for groups of order p^4*, Proc. Japan Acad. **77**, Ser. A (2001), 95-98 (jointly with L. R. Vermani).
- *On automorphisms of some p-groups*, Proc. Japan Acad. **78**, Ser. A (2002), 46-50 (jointly with L. R. Vermani).
- *On automorphisms of some finite p-groups*, Proc. Indian Acad. Sci. (Math. Sci.) **118** (2008), 1-11.
- *On Sha-rigidity of groups of order p^6*, J. Algebra **428** (2015), 26 - 42 (jointly with P. K. Rai).

We start with class-preserving automorphisms of unitriangular groups. Let $\operatorname{UT}_n(K)$ denote the unitriangular group consisting of all $n \times n$ unitriangular matrices having entries in a field K, where $n \geq 3$. Since the first example of a group G (Burnside group G isomorphic to the group W defined above) such that $\operatorname{Inn}(G) < \operatorname{Aut}_c(G)$ comes from unitriagular groups, it is quite natural to pose the following problems:

Problem A. Let $G = \mathrm{UT}_n(K)$. Find necessary and sufficient conditions on G such that $\mathrm{Inn}(G) = \mathrm{Aut}_c(G)$.
Problem B. Let $G = \mathrm{UT}_n(K)$ be such that $\mathrm{Inn}(G) < \mathrm{Aut}_c(G)$. Study the stucture of $\mathrm{Aut}_c(G)$.

We provide a solution to Problem A in the following theorem.

Theorem 1.13. *Let $G := \mathrm{UT}_n(K)$, where K is a field and $n \geq 3$. Then $\mathrm{Aut}_c(G) = \mathrm{Inn}(G)$ if and only if K is a prime field.*

On the lines of the proof of Theorem A, one can easily show that $\mathrm{Aut}_c(\mathrm{UT}_n(\mathbb{Z})) = \mathrm{Inn}(\mathrm{UT}_n(\mathbb{Z}))$ for every positive integer n, where $\mathbb{Z}$ denotes the ring of integers.

Next we study Problem B for the following special case. For integers $n \geq 3$, $m \geq 1$ and a prime p, let $\mathrm{UT}_n(\mathbb{F}_{p^m})$ denote the group of all $n \times n$ unitriangular matrices with entries in the field $\mathbb{F}_{p^m}$. Set $G_n^{(m)} := \mathrm{UT}_n(\mathbb{F}_{p^m})/\gamma_3(\mathrm{UT}_n(\mathbb{F}_{p^m}))$, where $\gamma_3(\mathrm{UT}_n(\mathbb{F}_{p^m}))$ denotes the third term in the lower central series of $\mathrm{UT}_n(\mathbb{F}_{p^m})$. In the following theorem, we calculate the group $\mathrm{Aut}_c(G_n^{(m)})$ and notice (surprisingly) that $|\mathrm{Aut}_c(G_n^{(m)})/\mathrm{Inn}(G_n^{(m)})|$ is independent of n.

Theorem 1.14. *Let $G_n^{(m)}$ be the group defined in the preceding paragraph. Then $\mathrm{Aut}_c(G_n^{(m)})$ is elementary abelian p-group of order $p^{2m^2+mn-3m}$. Moreover the order of $\mathrm{Aut}_c(G_n^{(m)})/\mathrm{Inn}(G_n^{(m)})$ is $p^{2m(m-1)}$, which is independent of n.*

We now consider p-groups of order at most p^6. As mentioned above, W. Burnside constructed a group G of order p^6, p an odd prime, such that $\mathrm{Out}_c(G) \neq 1$. In [106], [89] and [111] some more groups were constructed or identified such that $\mathrm{Out}_c(G) \neq 1$. But the order of all these groups is $\geq p^6$. Therefore, a natural question which arises here is the following:
Question. What is the smallest value of n such that there exists a p-group G of order p^n for which $\mathrm{Out}_c(G) \neq 1$?

We proved in the articles listed at the second, third and fourth place in the above list ([10, 11, 12]) the following results:

1. $\mathrm{Out}_c(G) = 1$ for all p-groups G having order less than or equal to p^4.
2. $\mathrm{Out}_c(G) = 1$ for all finite p-groups G having a maximal cyclic subgroup.
3. $\mathrm{Out}_c(G) = 1$ for all finite p-groups G having order p^n and such that it has a cyclic normal subgroup of order p^{n-2} (except two cases).
4. $\mathrm{Out}_c(G) = 1$ for all finite meta-cyclic groups G.
5. $\mathrm{Out}_c(G) \neq 1$ for the following finite 2-groups G having order 2^n and such that it has a cyclic normal subgroup of order 2^{n-2}:

$$\begin{aligned}
G &= \; < x,\ y,\ z \mid x^{2^{n-2}} = 1 = y^2 = z^2,\ y\,x\,y = x^{1+2^{n-3}}, \\
&\quad\; z\,x\,z = x^{-1+2^{n-3}},\ z\,y\,z = y >; \\
G &= \; < x,\ y,\ z \mid x^{2^{n-2}} = 1 = y^2 = z^2,\ y\,x\,y = x^{1+2^{n-3}}, \\
&\quad\; z\,x\,z = x^{-1+2^{n-3}},\ z\,y\,z = y\,x^{2^{n-3}} > .
\end{aligned}$$

So if we put $n = 5$, we get groups of order 2^5 for which $\mathrm{Out}_c(G) \neq 1$

In the fifth article ([24]) we proved the following useful criterion for class-preserving automorphisms of finite groups:

Theorem 1.15. *Let G and H be two finite non-abelian isoclinic groups. Then $\mathrm{Aut}_c(G) \cong \mathrm{Aut}_c(H)$.*

Using this theorem and classification of p-groups of order at most p^5 from [83], we prove the following result [24]:

Theorem 1.16. *Let G be a finite p-group of order p^5, where p is an odd prime. Then $\mathrm{Out}_c(G) \neq 1$ if and only if G is isoclinic to one of the following groups:*

$$\begin{aligned}
G = \phi_7(1^5) &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \beta | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \beta] = \alpha_3, \alpha^p = \alpha_1^{(p)} \\
&= \alpha_{i+1}^p = \beta^p = 1, i = 1, 2\rangle; \\
G = \phi_{10}(1^5) &= \langle \alpha, \alpha_1, \alpha_2, \alpha_3, \alpha_4 | [\alpha_i, \alpha] = \alpha_{i+1}, [\alpha_1, \alpha_2] = \alpha_4, \alpha^p = \alpha_1^{(p)} \\
&= \alpha_{i+1}^{(p)} = 1, i = 1, 2, 3\rangle,
\end{aligned}$$

where generators and relators have meaning in the sense of R. James [83].

As a consequence of all these results we answer the above mentioned question in the following:

Theorem 1.17. *The smallest value of n such that there exists a p-group G of order p^n for which $\mathrm{Out}_c(G) \neq 1$ is 5, where p is any prime, even or odd.*

We went a step further and classified all groups G of order p^6 with $\mathrm{Out}_c(G) \neq 1$, and computed the size of $\mathrm{Out}_c(G)$ for the groups G for which it is not 1. We used the classification of groups from [83] and then applied Theorem 1.15 to reduce the problem upto isoclinism. This is done in the last article listed above ([19]) proving the following result:

Theorem 1.18. *Let G be a group of order p^6 for an odd prime p. Then $Out_c(G) \neq 1$ if and only if G belongs to one of the isoclinism families Φ_k for $k = 7, 10, 13, 15, 18, 20, 21, 24, 30, 36, 38, 39$. Moreover,*

(1) if G belongs to one of the isoclinism families Φ_k for $k = 7, 10, 24, 30, 36, 38, 39$, then $|Out_c(G)| = p$,
(2) if G belongs to one of the isoclinism families Φ_k for $k = 13, 18, 20$, then $|Out_c(G)| = p^2$, and
(3) if G belongs to one of the isoclinism families Φ_k for $k = 15, 21$, then $|Out_c(G)| = p^4$.

2. Central automorphisms

Let G be a finite group. An automorphism α of G is called *central* if $x^{-1}\alpha(x) \in \mathrm{Z}(G)$ for all $x \in G$. The set of all central automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$. We denote this group by $\mathrm{Autcent}(G)$. Notice that $\mathrm{Autcent}(G) = \mathrm{C}_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$, the centralizer of $\mathrm{Inn}(G)$ in $\mathrm{Aut}(G)$, and $\mathrm{Autcent}(G) = \mathrm{Aut}(G)$ if $\mathrm{Aut}(G)$ is abelian. We denote the commutator and Frattini subgroup of G with $\gamma_2(G)$ and $\Phi(G)$, respectively. Let $G^p = \langle x^p \mid x \in G\rangle$ and $G_p = \langle x \in G \mid x^p = 1\rangle$. For finite abelian groups H and K, $\mathrm{Hom}(H, K)$ denotes the group of all homomorphisms from H to K. Throughout the section, p always denotes an odd prime.

2.1. Miller p-groups.

This subsection includes the following articles:

- *On finite p-groups whose automorphisms are all central*, Israel J. Maths. **189** (2012), 225-236 (jointly with V. K. Jain).
- *Finite p-groups with abelian automorphism group*, Internat. J. Algebra Comput., **23** (2013) 1063-1077 (jointly with V. K. Jain and P. K. Rai).
- *Note on Caranti's method of construction of Miller groups,* Monatsh. Math. **185** (2018), 87-101 (jointly with Rahul D. Kitture).

In 1908, H. Hilton [74, p 233] asked the following question: Whether a non-abelian group can have an abelian group of isomorphisms (automorphisms). An affirmative answer to this question was given by G. A. Miller [96] in 1913. He constructed a non-abelian group G of order 64 such that $\mathrm{Aut}(G)$ is abelian and has order 128. More examples of such finite 2-groups were constructed by R. R. Struik [110] in 1982, M. J. Curran [51] in 1987 and A. Jamali [81] in 2002. In 1974, H. Heineken and H. Liebeck [71] showed that for any finite group K, there exists a finite p-group G such that $\mathrm{Aut}(G)/\mathrm{Autcent}(G)$ is isomorphic to K. In particular, for $K = 1$, this provides a p-group G such that $\mathrm{Aut}(G) = \mathrm{Autcent}(G)$ is an elementary abelian group. In 1975, D. Jonah and M. Konvisser [85] constructed 4-generated groups of order p^8 such that $\mathrm{Aut}(G) = \mathrm{Autcent}(G)$ and $\mathrm{Aut}(G)$ is an elementary abelian group of order p^{16}, where p is any prime.

In 1927, C. Hopkins [75] proved, among other things, that if G is a group such that $\mathrm{Aut}(G)$ is abelian, then G can not have a non-trivial abelian direct factor. But this result is not true for 2-groups, as proved by B. Earnley in his thesis [59, Theorem 2.3] in 1975. Among other things, Earnley proved (i) there is no group G of order p^5 such that $\mathrm{Aut}(G)$ is abelian, (ii) for each positive integer $n \geq 4$, there exist n-generated p-groups G such that $\mathrm{Aut}(G)$ is abelian. On the way to constructing finite p-groups of class 2 such that all normal subgroups of G are characteristic, in 1979 H. Heineken [72] produced groups G such that $\mathrm{Aut}(G)$ is abelian. In 1994, M. Morigi [98] proved that there exists no group of order p^6 whose group of automorphisms is abelian and constructed groups G of order p^{n^2+3n+3} such that $\mathrm{Aut}(G)$ is abelian, where n is a positive integer. In particular, for $n = 1$, it provides a group of order p^7 having an abelian automorphism group. Further in 1995, M. Morigi [99] proved that the minimal number of generators for a p-group with abelian automorphism group is 4. In 1995, P. Hegarty [70] proved that if G is a non-abelian p-group such that $\mathrm{Aut}(G)$ is abelian, then $|\mathrm{Aut}(G)| \geq p^{12}$, and the minimum is obtained by the group of order p^7 constructed by M. Morigi. Moreover, in 1998 G. Ban and S. Yu [36] obtained independently the same result and proved that if G is a group of order

p^7 such that $\mathrm{Aut}(G)$ is abelian, then $|\mathrm{Aut}(G)| = p^{12}$ (the last result is true for all primes, not only for p odd).

We would like to remark here that all the examples mentioned above are special p-groups, where p is an odd prime. Upto that point of time, no non-special p-group G was known such that $\mathrm{Aut}(G)$ is abelian. Our last statement is supported by the following conjecture of A. Mahalanobis [94]:

Conjecture. *For an odd prime p, let G be a finite p-group such that $\mathrm{Aut}(G)$ is abelian. Then G is a special p-group.*

We construct a family of counterexamples to this conjecture in the following theorem.

Theorem 2.1. *Let $m = n + 5$ and p be an odd prime, where n is a positive integer greater than or equal to 3. Then there exists a 4-generated group G of order p^m and exponent p^n such that $\mathrm{Aut}(G)$ is abelian, but G is not special. Moreover, $|\mathrm{Aut}(G)| = p^{n+10}$.*

We deviate for while from Miller groups to consider finite p-groups G for which $\mathrm{Aut}(G) = \mathrm{Autcent}(G)$ is non-abelian. In 1982, M. J. Curran [50] constructed groups G of order 2^7 such that $\mathrm{Aut}(G) = \mathrm{Autcent}(G)$ is non-abelian. Further, in 1984, J. J. Malone [91] constructed p-groups for odd primes such that $\mathrm{Aut}(G) = \mathrm{Autcent}(G)$ is non-abelian. We would like to remark here that the groups of Curran and Malone have direct factors. More precisely these groups were constructed by taking direct products of abelian (cyclic) p-groups and groups G such that $\mathrm{Aut}(G)$ is abelian. Examples of 2-groups G such that G does not have an abelian direct factor and $\mathrm{Aut}(G) = \mathrm{Autcent}(G)$ is non-abelian were constructed by S. P. Glasby [67] in 1986. Until recently, no examples of such p-groups were known (to the best of our knowledge) for an odd prime p. Our last statement is supported by the following problem of I. Malinowska [90, Problem 13].

Problem. *For an odd prime p, find a p-group G which has no non-trivial abelian direct factor and $\mathrm{Aut}(G) = \mathrm{Autcent}(G)$ is non-abelian.*

We construct examples of such groups in the following theorem.

Theorem 2.2. *Let $m = n + 7$ and p be an odd prime, where n is a positive integer greater than or equal to 3. Then there exists a group G of order p^m, exponent p^n and with no non-trivial abelian direct factor such that $\mathrm{Aut}(G) = \mathrm{Autcent}(G)$ is non-abelian.*

Two obvious weaker forms of the conjecture of Mahalanobis are: (WC1) For a finite p-group G with $\mathrm{Aut}(G)$ abelian, $\mathrm{Z}(G) = \Phi(G)$ always holds true; (WC2) For a finite p-group G with $\mathrm{Aut}(G)$ abelian and $\mathrm{Z}(G) \neq \Phi(G)$, $\gamma_2(G) = \mathrm{Z}(G)$ always holds true. So, on the way to exploring some general structure on the class of such groups G, it is natural to ask the following question:

Question. Does there exist a finite p-group G such that $\gamma_2(G) \leq \mathrm{Z}(G) < \Phi(G)$ and $\mathrm{Aut}(G)$ is abelian?

Disproving (WC1) and (WC2), we provide affirmative answer to this question, in the following theorem:

Theorem 2.3. *For every positive integer $n \geq 4$ and every odd prime p, there exists a group G of order p^{n+10} and exponent p^n such that*

(1) for $n = 4$, $\gamma_2(G) = \mathrm{Z}(G) < \Phi(G)$ and $\mathrm{Aut}(G)$ is abelian;

(2) for $n \geq 5$, $\gamma_2(G) < \mathrm{Z}(G) < \Phi(G)$ and $\mathrm{Aut}(G)$ is abelian.

Moreover, the order of $\mathrm{Aut}(G)$ is p^{n+20}.

One more weaker form of the above said conjecture is: (WC3) If $\mathrm{Aut}(G)$ is an elementary abelian p-group, then G is special. As remarked above, all p-groups G (except the ones in [6]) available in the litrature and having abelian automorphism group are special p-groups. Thus it follows that $\mathrm{Aut}(G)$, for all such groups G, is elementary abelian. Y. Berkovich and Z. Janko [37, Problem 722] published the following long standing problem: *(Old problem) Study the p-groups G with elementary abelian* $\mathrm{Aut}(G)$. For such groups we prove

Theorem 2.4. *Let G be a finite p-group such that $\mathrm{Aut}(G)$ is elementary abelian, where p is an odd prime. Then one of the following two conditions holds true:*

(1) $\mathrm{Z}(G) = \Phi(G)$ is elementary abelian;

(2) $\gamma_2(G) = \Phi(G)$ is elementary abelian.

Moreover, the exponent of G is p^2.

Let G be an arbitrary finite p-group such that $\mathrm{Aut}(G)$ is elementary abelian. Then it follows from the Theorem 2.4 that one of the following two conditions necessarily holds true: (C1) $\mathrm{Z}(G) = \Phi(G)$ is elementary abelian; (C2) $\gamma_2(G) = \Phi(G)$ is elementary abelian. *So one might expect that for such groups G both of the conditions (C1) and (C2) hold true, i.e., WC(3) holds true, or, a little less ambitiously, (C1) always holds true or (C2) always holds true.* In the following two theorems we show that none of the statements in the preceding sentence hold true.

Theorem 2.5. *There exists a group G of order p^9 such that $\mathrm{Aut}(G)$ is elementary abelian, $\Phi(G) < \mathrm{Z}(G)$ and $\gamma_2(G) = \Phi(G)$ is elementary abelian. Moreover, the order of $\mathrm{Aut}(G)$ is p^{20}.*

Theorem 2.6. *There exists a group G of order p^8 such that $\mathrm{Aut}(G)$ is elementary abelian, $\gamma_2(G) < \Phi(G)$ and $\mathrm{Z}(G) = \Phi(G)$ is elementary abelian. Moreover, the order of $\mathrm{Aut}(G)$ is p^{16}.*

The construction of the Miller groups mentioned above uses ad-hoc methods and tedious computations. Concerning these difficulties in the construction, A. Caranti [49, §5, §6], among other things, provided two methods to construct non-special Miller p-group from a special Miller p-group. The methods are interesting as these involve simple and elegant module theoretic arguments, instead of cumbersome computations. But, unfortunately there remained a gap in the proof. We attempted to fill up this gap in one direction which was motivated by an observation from Theorem 2.7 and 2.8 below.

According to the methods in [49], given a special Miller p-group H and a cyclic p-group $\langle z \rangle$ of order $\geq p^2$, a group $G := H \rtimes_M \langle z \rangle$ is constructed as *amalgamated semi-direct product* of H by $\langle z \rangle$ (amalgamated) over a subgroup $M \leq H'$ of order p (see [60, p.27] or §2 for the definition). With appropriate action of z on H and some conditions on H, it was claimed in [49] that for every choice of M (of order p) in H' but not in H^p, G is a Miller group. We show that this is not always true.

Let $H = \langle a, b, c, d \rangle$ be a p-group of class 2 with the following additional relations:

$$a^p = [a, c], \quad b^p = [a, bcd], \quad c^p = [b, cd], \quad d^p = [b, d].$$

The group H is a special Miller p-group of order p^{10}. Note that $\langle [a, b] \rangle$ and $\langle [a, d] \rangle$ are subgroups of order p in H' but not in H^p.

Theorem 2.7. *Let H be the special Miller p-group as above, $\langle z \rangle$ a cyclic group of order p^2 and $M \leq H'$ a subgroup of order p with $M \nsubseteq H^p$. Let $G_1 = H \rtimes_M \langle z \rangle$, where z acts trivially on H. Then G_1 is a non-special p-group and the following holds true:*

(1) If $M = \langle [a, b] \rangle$, then G_1 is a Miller group.
(2) If $M = \langle [a, d] \rangle$, then G_1 is not a Miller group.

Theorem 2.8. *Let H be the special Miller p-group as above, $\langle w \rangle$ a cyclic group of order p^3 and $M \leq H'$ a subgroup of order p with $M \nsubseteq H^p$. Let $G_2 = H \rtimes_M \langle w \rangle$, where w normalizes H via the following non-inner central automorphism of H:*

$$waw^{-1} = ad^p, \quad wbw^{-1} = b, \quad wcw^{-1} = c, \quad wdw^{-1} = d.$$

Then G_2 is a non-special p-group of order p^{12} and the following holds true:

(1) If $M = \langle [a, b] \rangle$, then G_2 is a Miller group.
(2) If $M = \langle [a, d] \rangle$, then G_2 is not a Miller group.

Note that, although part (2) of the above theorems provides counter-examples for the two methods, part (1) motivates to find a condition on the choice of M which would imply that G is a Miller group. We provide a sufficient condition on the choice of M for which the methods work. This condition is stated in the following theorems, for which we set some common hypotheses.

(i) Let H be a special Miller p-group such that $H^p < H'$ and the map $H/H' \to H^p$, $hH' \mapsto h^p$ is injective.

(ii) Let M be a subgroup of order p in H' but not in H^p.

Theorem 2.9. *With H and M as in (i) and (ii) above, let $G_1 = H \rtimes_M \langle z \rangle$, where z acts trivially on H, $o(z) = p^2$. If H/M is a special Miller p-group, then G_1 is a (non-special) Miller group.*

Let H also satisfy the following condition:

(iii) H' is freely generated by $[x_i, x_j]$, $1 \leq i < j \leq n$, provided $\{x_1, \ldots, x_n\}$ is a minimal generating set for H.

Theorem 2.10. *With H and M as in (i) to (iii) above, let $G_2 = H \rtimes_M \langle w \rangle$, where $o(w) > p^2$ and w acts on H via a non-inner central automorphism of H. If H/M is a special Miller p-group, then G_2 is a (non-special) Miller group.*

2.2. **Other results on central automorphisms.** Contains results from the following articles.

- *On finite p-groups whose central automorphisms are all class preserving*, Comm. Algebra, Vol 41, No. 12 (2013), 4576 - 4592.
- *On central automorphisms fixing the centrer elementwise*, Comm. Algebra **37** (2009), 4325-4331.

In 1999, A. Mann [92, Question 10] asked the following question: *Do all p-groups have automorphisms that are not class-preserving? If the answer is no, which are the groups that have only class-preserving automorphisms?* The first part of the question have a negative answer. The examples of finite p-groups G such that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$ are already known in the literature. Such groups G, having nilpotency class 2 were constructed by H. Heineken [72] in 1980 and that having nilpotency class 3 were constructed by I. Malinowska [89] in 1992. So the second part of the question of Mann becomes relevant. Let us modify the question of Mann to make it more precise in the present scenario.

Question. Let $n \geq 4$ be a positive integer and p be a prime number. Does there exist a finite p-group of nilpotency class n such that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$.

The second part of Mann's question, which clearly talks about the classification, can be stated as

Problem (A. Mann). Study (Classify) finite p-groups G such that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$.

We consider this problem for p-groups of nilpotency class 2 and make the stone rolling. Notice that, for a finite p-group of class 2, we have the following sequence of subgroups

$$\mathrm{Aut}_c(G) \leq \mathrm{Autcent}(G) \leq \mathrm{Aut}(G).$$

The groups G such that $\mathrm{Autcent}(G) = \mathrm{Aut}(G)$, have been studied, every now and then, by many mathematicians (see [6] and [7] for recent developments and other references). So for studying groups G of nilpotency class 2 such that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$, one needs to concentrate on the groups G satisfying

Hypothesis A. $\mathrm{Aut}_c(G) = \mathrm{Autcent}(G)$.

For a finite p-group G, we denote by $\Omega_m(G)$ the subgroup $\langle x \in G \mid x^{p^m} = 1 \rangle$. Notice that the nilpotency class of a non-abelian finite p-group satisfying Hypothesis A is 2. Let G be a finite p-group of class 2. Then $G/\mathrm{Z}(G)$ is abelian. Consider the following cyclic decomposition of $G/\mathrm{Z}(G)$.

$$G/\mathrm{Z}(G) = \mathrm{C}_{p^{m_1}} \times \cdots \times \mathrm{C}_{p^{m_d}}$$

such that $m_1 \geq m_2 \geq \cdots \geq m_d \geq 1$, where $\mathrm{C}_{p^{m_i}}$ denotes the cyclic group of order p^{m_i} for $1 \leq i \leq d$. The integers $p^{m_1}, \ldots, p^{m_d}$ are unique for $G/\mathrm{Z}(G)$ and these are called the *invariants* of $G/\mathrm{Z}(G)$. Now we state our first result

Theorem 2.11. *Let G be a finite p-group of class 2 and $p^{m_1}, \ldots, p^{m_d}$ be the invariants of $G/\mathrm{Z}(G)$. Then G satisfies Hypothesis A if and only if $\gamma_2(G) = \mathrm{Z}(G)$ and $|\mathrm{Aut}_c(G)| = \Pi_{i=1}^{d} |\Omega_{m_i}(\gamma_2(G))|$.*

Our next result is

Theorem 2.12. *Let G be a finite p-group of class 2 satisfying Hypothesis A. Then G is minimally generated by even number of elements.*

For finite p-groups whose automorphisms are all class preserving, we prove

Theorem 2.13. *Let G be a non-abelian finite p-group such that $\mathrm{Aut}(G) = \mathrm{Aut}_c(G)$, where p is an odd prime. Then the following statements hold true.*

(2.14a) $\gamma_2(G)$ *cannot be cyclic.*

(2.14b) *If* $\mathrm{Aut}(G)$ *is elementary abelian, then* G *is a Camina special* p*-group.*

(2.14c) *If* $\mathrm{Aut}(G)$ *is abelian, then* $d(G)$ *is even.*

(2.14d) *If* $\mathrm{Aut}(G)$ *is abelian, then* $|G| \geq p^8$ *and* $|\mathrm{Aut}(G)| \geq p^{12}$.

(2.14e) *With* $\mathrm{Aut}(G)$ *abelian,* $|\mathrm{Aut}(G)| = p^{12}$ *if and only if* $|G| = p^8$.

(2.14f) *If* $\mathrm{Aut}(G)$ *is abelian of order* p^{12}*, then* $\mathrm{Aut}(G)$ *is elementary abelian*

(2.14g) *There exists a group* G *of order* 3^8 *such that* $|\mathrm{Aut}(G)| = |\mathrm{Aut}_c(G)| = 3^{12}$.

M. S. Attar Attar [35] proved for a finite p-group G that all central automorphisms of G fixing its center elementwise are inner if and only if G is abelian or G is nilpotent of class 2 and $\mathrm{Z}(G)$ is cyclic. We study the necessary and sufficient conditions on a finite p-group G of class 2 whose all central automorphisms fix the center of it elementwise.

Let G be a finite p-group of class 2. Then $G/\mathrm{Z}(G)$ and $\gamma_2(G)$ have equal exponent p^c (say). Let

$$G/\mathrm{Z}(G) = \mathrm{C}_{p^{a_1}} \times \mathrm{C}_{p^{a_2}} \times \cdots \times \mathrm{C}_{p^{a_r}},$$

where $\mathrm{C}_{p^{a_i}}$ is a cyclic group of order p^{a_i}, $1 \leq i \leq r$, and $a_1 \geq a_2 \geq \cdots \geq a_r > 0$. Let k be the largest integer between 1 and r such that $a_1 = a_2 = \cdots = a_k = c$. Notice that $k \geq 2$. Set $\bar{\mathrm{M}} = \mathrm{M}/\mathrm{Z}(G) = \mathrm{C}_{p^{a_1}} \times \cdots \times \mathrm{C}_{p^{a_k}}$. Let

$$G/\gamma_2(G) = \mathrm{C}_{p^{b_1}} \times \mathrm{C}_{p^{b_2}} \times \cdots \times \mathrm{C}_{p^{b_s}},$$

where $b_1 \geq b_2 \geq \cdots \geq b_s > 0$, be a cyclic decomposition of $G/\gamma_2(G)$ such that $\bar{\mathrm{M}}$ is isomorphic to a subgroup of $\bar{\mathrm{N}} = \mathrm{N}/\gamma_2(G) := \mathrm{C}_{p^{b_1}} \times \mathrm{C}_{p^{b_2}} \times \cdots \times \mathrm{C}_{p^{b_k}}$. Using the above terminology we state our result in

Theorem 2.15. *Let G be a finite p-group of nilpotency class 2. Then all central automorphisms of G fix the center of G elementwise if and only if $r = s$, $(G/\mathrm{Z}(G))/\bar{\mathrm{M}} \cong (G/\gamma_2(G))/\bar{\mathrm{N}}$ and the exponents of $\mathrm{Z}(G)$ and $\gamma_2(G)$ are equal.*

3. Automorphisms of groups

In this section we include results from the following articles.

- *Automorphisms of abelian group extensions*, J. Algebra **324** (2010), 820-830 (jointly with I. B. S. Passi and Mahender Singh).
- *Non-inner automorphisms of order p in finite p-groups of coclass 3*, Monatsh. Math. **183** (2017), 679 - 697 (jointly with Marco Ruscitti, Leire Legarreta).
- *On automorphisms of finite p-groups*, J. Group Theory **10** (2007), 859-866.

We start with extensions and lifting of automorphisms. Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be a short exact sequence. We construct certain exact sequences, similar to the one due to C. Wells [114], and apply them to study extensions and liftings of automorphisms in abelian extensions. More precisely, we study, for abelian extensions, the following well-known problem (see [45, 84, 104]):

Problem. *Let N be a normal subgroup of G. Under what conditions (i) can an automorphism of N be extended to an automorphism of G; (ii) an automorphism of G/N is induced by an automorphism of G?*

Let $1 \to N \to G \xrightarrow{\pi} H \to 1$ be an abelian extension. We fix a left transversal $t : H \to G$ for H in G such that $t(1) = 1$, so that every element of G can be written uniquely as $t(x)n$ for some $x \in H$ and $n \in N$. Given an element $x \in H$, we define an action of x on N by setting $n^x = t(x)^{-1}nt(x)$; we thus have a homomorphism $\alpha : H \to \mathrm{Aut}(N)$ enabling us to view N as a right H-module.

A pair $(\theta, \phi) \in \mathrm{Aut}(N) \times \mathrm{Aut}(H)$ is called *compatible* if $\theta^{-1}x^{\alpha}\theta = (x^{\phi})^{\alpha}$ for all $x \in H$. Let C denote the group of all compatible pairs. Let $\mathrm{C}_1 = \{\theta \in \mathrm{Aut}(N) \mid (\theta, 1) \in \mathrm{C}\}$ and $\mathrm{C}_2 = \{\phi \in \mathrm{Aut}(H) \mid (1, \phi) \in \mathrm{C}\}$.

We denote by $\mathrm{Aut}^{N,H}(G)$ the group of all automorphisms of G which centralize N (i.e., fix N element-wise) and induce identity on H. By $\mathrm{Aut}^{N}(G)$ and $\mathrm{Aut}_N(G)$ we denote respectively the group of all automorphisms of G which centralize N and the group of all automorphisms α of G which normalize N (i.e., $\alpha(N) = N$). By $\mathrm{Aut}_N^H(G)$ we denote the group of all automorphisms of G which normalize N and induce identity on H.

Observe that an automorphism $\gamma \in \mathrm{Aut}_N(G)$ induces automorphisms $\theta \in \mathrm{Aut}(N)$ and $\phi \in \mathrm{Aut}(H)$ given by $\theta(n) = \gamma(n)$ for all $n \in N$ and $\phi(x) = \gamma(t(x))N$ for all $x \in H$. We can thus define a homomorphism $\tau : \mathrm{Aut}_N(G) \to \mathrm{Aut}(N) \times \mathrm{Aut}(H)$ by setting $\tau(\gamma) = (\theta, \phi)$. We denote the restrictions of τ to $\mathrm{Aut}_N^H(G)$ and $\mathrm{Aut}^N(G)$ by τ_1 and τ_2 respectively.

With the above notation, there exists the following exact sequence, first constructed by C. Wells [114], which relates automorphisms of group extensions with group cohomology:

$$1 \to \mathrm{Z}^1_\alpha(H, \mathrm{Z}(N)) \to \mathrm{Aut}_N(G) \xrightarrow{\tau} C \to \mathrm{H}^2_\alpha(H, \mathrm{Z}(N)).$$

Recently P. Jin [84] gave an explicit description of this sequence for automorphisms of G inducing identity on H and obtained some interesting results regarding extensions of automorphisms of N to automorphisms of G inducing identity on H. We continue in the present work this line of investigation.

By establishing suitable exact sequences, which are more or less special cases of Wells' exact sequence in our settings, we reduce the problem of lifting of automorphisms of H to G to the problem of lifting of automorphisms of Sylow subgroups of H to automorphisms of their pre-images in G, and prove the following result:

Theorem 3.1. *Let N be an abelian normal subgroup of a finite group G. Then the following hold:*

(1) *An automorphism ϕ of G/N lifts to an automorphism of G centralizing N provided the restriction of ϕ to some Sylow p-subgroup P/N of G/N, for each prime number p dividing $|G/N|$, lifts to an automorphism of P centralizing N.*

(2) *If an automorphism ϕ of G/N lifts to an automorphism of G centralizing N, then the restriction of ϕ to a characteristic subgroup P/N of G/N lifts to an automorphism of P centralizing N.*

We mention below two corollaries to illustrate this theorem. These corollaries show that, in many cases, the hypothesis of the above theorem is naturally satisfied.

Corollary 3.2. *Let N be an abelian normal subgroup of a finite group G such that G/N is nilpotent. Then an automorphism ϕ of G/N lifts to an automorphism of G centralizing N if, and only if the restriction of ϕ to each Sylow subgroup P/N of G/N lifts to an automorphism of P centralizing N.*

An automorphism α of a group G is said to be *commuting automorphism* if $x\alpha(x) = \alpha(x)x$ for all $x \in G$. It follows from a result of Deaconescu, Silberberg and Walls [58, Remark 4.2] that each Sylow subgroup of a finite group G is kept invariant by every commuting automorphism of G. Thus we have the following result:

Corollary 3.3. *Let N be an abelian normal subgroup of a finite group G. Then a commuting automorphism ϕ of G/N lifts to an automorphism of G centralizing N if, and only if the restriction of ϕ to each Sylow subgroup P/N of G/N lifts to an automorphism of P centralizing N.*

The motive of the following piece of work is to contribute to the following longstanding conjecture of Berkovich [86, Problem 4.13], posed in 1973:

Conjecture. Every finite p-group admits a non-inner automorphism of order p, where p denotes a prime number.

This conjecture, which will be called *the conjecture* throughout this discussion, can be viewed as a refinement of the following celebrated theorem of Gaschütz [65]: Every non-abelian finite p-group admits a non-inner automorphism of order some power of p.

The conjecture has attracted the attention of many mathematicians during the last couple of decades, and has been confirmed for many interesting classes of finite p-groups. It is interesting to put on record that, in 1965, Liebeck [87] proved the existence of a non-inner automorphism of order p in all finite p-groups of class 2, where p is an odd prime. For $p = 2$, he proved the existence of a non-inner automorphism of order 2 or 4. The fact that there always exists a non-inner automorphism of order 2 in all finite 2-groups of class 2 was proved by Abdollahi [31] in 2007. The conjecture was confirmed for finite regular p-groups by Schmid [107] in 1980. Deaconescu [57] proved it for all finite p-groups G which are not strongly Frattinian, in other words, groups satisfying $C_G(Z(\Phi(G))) \neq \Phi(G)$.

Abdollahi [32] proved it for finite p-groups G such that $G/Z(G)$ is a powerful p-group, and Jamali and Viseh [82] proved the conjecture for finite p-groups with cyclic commutator subgroup. In the realm of finite groups, quite recently, the conjecture has been confirmed for p-groups of nilpotency class 3, by Abdollahi, Ghoraishi and Wilkens [33], and for p-groups of coclass 2 by Abdollahi et al [34]. Finally, for semi-abelian p-groups, the conjecture has been confirmed by Benmoussa and Guerboussa [38].

We add an important class of p-groups to the above list by proving the following result.

Main Theorem. *The conjecture holds true for all non-abelian finite p-groups of coclass* 3*, where p is a prime integer such that $p \neq 3$.*

Now we mention our work done in the third article. Let G be a finite p-group and N be a non-trivial proper normal subgroup of G. (G, N) is called a *Camina pair* if $xN \subseteq x^G$ for all $x \in G - N$. It follows that (G, N) is a Camina pair if and only if $N \subseteq [x, G]$ for all $x \in G - N$, where $[x, G] = \{[x, g] | g \in G\}$. We prove the following theorem:

Theorem 3.4. *Let G be a finite p-group such that $(G, \mathrm{Z}(G))$ is a Camina pair. Then $|G|$ divides $|\operatorname{Aut}(G)|$.*

This theorem extends the known classes of finite p-groups for which the following well known conjecture [95, Problem 12.77] holds true:

Conjecture. Let G be a non-cyclic p-group of order p^n, where $n \geq 3$. Then $|G|$ divides $|\operatorname{Aut}(G)|$.

This conjecture has been established for p-groups of class 2 [64], for p-abelian p-groups [55], for abelian p-groups and p-groups of maximal class [102], for p-groups of order $\leq p^7$ [56, 63, 66], for finite modular p-groups [54] and for some other classes of finite p-groups.

Remark. The existence of finite p-groups for which this conjecture fails was estabilshed by Gonzalez-Sanchez and Jaikin-Zapirain [68] in 2015. For more details and elaborated proofs, please see our recent book [18].

4. Conjugacy classes in finite groups

Study of conjugacy classes in finite groups is an active and interesting area of research. A kind of general question is the following:

Question. How the structure of conjugacy classes of finite groups reflects the structure of the group and vice versa?

4.1. **Product of conjugacy classes.** Results from the article

– *Finite groups with many product conjugacy classes*, Israel J. Math. **154** (2006), 29-49 (jointly with E. C. Dade).

We consider finite groups G in which the product of two conjugacy classes is itself a conjugacy class, except in a few cases where that product obviously can't be a single conjugacy class. If we denote by x^G the G-conjugacy class of an element $x \in G$, then the product $x^G y^G$ of two conjugacy classes x^G and y^G in G is itself a conjugacy class if and only if it satisfies

$$x^G y^G = (xy)^G. \tag{4.1}$$

This equation holds trivially if either x or y belongs to the center $\mathrm{Z}(G)$ of G. So we may assume that both x and y lie in the complementary subset $G - \mathrm{Z}(G)$ to $\mathrm{Z}(G)$ in G. Then (4.1) certainly fails when x^G is the inverse class $(y^{-1})^G$ to y^G, since $x^G y^G$ then contains the trivial conjugacy class $1^G = \{1\}$, but has size $|x^G y^G| \geq |x^G| > 1$, and thus must contain at least one other conjugacy class. Our first assumption is that this is the only situation in which (4.1) fails. So we are going to consider finite groups G satisfying

Hypothesis 4.2. *Equation* (4.1) *holds for all $x, y \in G$ such that $x^G \neq (y^{-1})^G$.*

Our first main result is the following:

Theorem 4.3. *A finite group G satisfies Hypothesis 5.2 if and only if it is isomorphic to one of the groups in the following list:*

(4.4a) *Any finite abelian group.*

(4.4b) *A non-abelian Camina p-group, for some prime p.*

(4.4c) *The group $F^+ \rtimes F^\times$, for some finite field F with $|F| > 2$.*

(4.4d) *The group $E_9 \rtimes Q_8$,*

where E_9 is the elementary abelian group of order 9.

Another situation where (4.1) clearly fails is when the classes x^G and $(y^{-1})^G$ have the same non-trivial image $x^G\,\mathrm{Z}(G)/\,\mathrm{Z}(G) = (y^{-1})^G\,\mathrm{Z}(G)/\,\mathrm{Z}(G) \neq 1_{G/\,\mathrm{Z}(G)}$ in the factor group $G/\,\mathrm{Z}(G)$. In this case $x^G y^G$ contains some element $z \in \mathrm{Z}(G)$, yet has size $|x^G y^G| \geq |x^G| > 1$. So it contains both $z^G = \{z\}$ and at least one other conjugacy class of G. Weakening Hypothesis A to avoid this situation, we obtain

Hypothesis 4.5. *If $x, y \in G$ and $x^G\,\mathrm{Z}(G) \neq (y^{-1})^G\,\mathrm{Z}(G)$, then (4.1) holds.*

When this weaker hypothesis holds for a finite group G, we show that it also holds for any finite group H isoclinic to G, in the sense of Philip Hall [69]. Our other main result is that this is the only freedom we have.

Theorem 4.6. *A finite group G satisfies Hypothesis 5.5 if and only if it is isoclinic to a group satisfying Hypothesis 5.2, and thus to one of the groups in the list (4.4).*

4.2. **Groups with only two conjugacy class sizes.** This subsection includes results from the following articles.

- *Finite p-groups of conjugate type $\{1, p^3\}$*, J. Group Theory **21** (2018), 65-82 (jointly with Tushar K. Naik).
- *Finite p-groups of conjugate rank one and nilpotency class 3,* Israel J. Math. (24 pages, accepted for publication, jointly with Rahul D. Kitture and Tushar K. Naik).

A finite group G is said to be of *conjugate type* $(1 = m_1, m_2, \cdots, m_r)$, $m_i < m_{i+1}$, if m_i's are precisely the different sizes of conjugacy classes of G. In this paper we restrict our attention on finite groups of conjugate type $(1, m)$. Investigation on such groups G was initiated by N. Ito [79] in 1953. He proved that groups of conjugate type $(1, m)$ are nilpotent, with m a prime power, say p^n. In particular, G is a direct product of its Sylow-p subgroup and some abelian p'-subgroup. So, it is sufficient to study finite *p-groups* of conjugate type $(1, p^n)$ for p a prime and $n \geq 1$ an integer. It was proved by K. Ishikawa [78] that the nilpotency class of such finite p-groups is either 2 or 3.

It follows that any two isoclinic groups are of same conjugate type. The study of finite p-groups of conjugate type $(1, p^n)$, up to isoclinism, was initiated by Ishikawa [77]. He classified such groups for $n \leq 2$. As a consequence, it follows that there is no finite p-group of nilpotency class 3 and conjugate type $(1, p)$, and there is a unique finite p-group, up to isoclinism, of nilpotency class 3 and conjugate type $(1, p^2)$.

For any positive integer $r \geq 1$ and prime $p > 2$, consider the following group constructed by Ito [79].

$$
\begin{aligned}
G_r \quad = \quad & \big\langle a_1, \ldots, a_{r+1} \mid [a_i, a_j] = b_{ij}, [a_k, b_{ij}] = 1, \\
& a_i^p = a_{r+1}^p = b_{ij}^p = 1, 1 \leq i < j \leq r+1, 1 \leq k \leq r+1 \big\rangle.
\end{aligned} \tag{4.7}
$$

It follows from [79, Example 1] that the group G_r defined in (4.7) is a special p-group of order $p^{(r+1)(r+2)/2}$ and exponent p, and $|G_r'| = p^{r(r+1)/2}$. This group has only two different conjugacy class sizes, namely 1 and p^r. Thus G_3 is of conjugate type $\{1, p^3\}$. For simplicity of notation, we assume that G_3 is generated by a, b, c and d.

In the following theorem we provide a classification of all finite p-groups of conjugate type $\{1, p^3\}$, $p > 2$, upto isoclinism.

Theorem 4.8. *Let G be a finite p-group of conjugate type $\{1, p^3\}$, $p > 2$. Then the nilpotency class of G is 2 and G is isoclinic to one of following groups:*

(i) A finite Camina p-group of nilpotency class 2 with commutator subgroup of order p^3;

(ii) The group G_3, defined in (4.7) for $r = 3$;

(iii) The quotient group G_3/M, where M is a normal subgroup of G_3 given by $M = \langle [a,\ b][c,\ d] \rangle$;

(iv) The quotient group G_3/N, where $N = \langle [a,\ b][c,\ d],\ [a,\ c][b,\ d]^t \rangle$ is a normal subgroup of G_3 with t any fixed integer non-square modulo p.

A similar theorem for $p = 2$ is also proved.

The examples of p-groups of nilpotency class 3 and of conjugate type $(1, p^{2m})$ are known for all $m \geq 1$. These examples appeared in the construction of certain Camina p-groups of nilpotency class 3 by Dark and Scoppola [53, p. 796-797]. It can be shown that for a given integer $m \geq 1$ and a prime $p > 2$, the p-group of conjugate type $(1, p^{2m})$ and nilpotency class 3, constructed by Dark and Scoppola, is isomorphic to $\mathcal{H}_m/\operatorname{Z}(\mathcal{H}_m)$, where $\mathcal{H}_m$ is presented as follows:

$$\mathcal{H}_m = \left\{ \begin{bmatrix} 1 & & & & \\ a & 1 & & & \\ c & b & 1 & & \\ d & ab-c & a & 1 & \\ f & e & c & b & 1 \end{bmatrix} : a, b, c, d, e, f \in \mathbb{F}_{p^m} \right\}. \tag{4.9}$$

In view of these examples, the question asked in [13] reduces to the following: Does there exist a finite p-group of nilpotency class 3 and of conjugate type $(1, p^n)$, for an odd prime p and odd integer $n \geq 5$?

We answer this question, by proving the following much more general result.

Theorem 4.10. *Let $p > 2$ be a prime and $n \geq 1$ an integer. Then there exist finite p-groups of nilpotency class 3 and conjugate type $(1, p^n)$ if and only if n is even. For each positive even integer $n = 2m$, every finite p-group of nilpotency class 3 and of conjugate type $(1, p^n)$ is isoclinic to the group $\mathcal{H}_m/\operatorname{Z}(\mathcal{H}_m)$, where $\mathcal{H}_m$ is as in (4.9).*

4.3. **Small conjugacy classes.**

– *On subgroup generated by small classes in finite groups*, Comm. Algebra **41** (2013), 3350-3354.

Let G be an arbitrary finite group. An element x of G is said to be *small* if x is non-central and lies in some conjugacy class of the smallest size in G. Let $M(G)$ denote the subgroup of G which is generated by all small elements of G.

In 1953, N. Ito [79] proved that if G is a finite group in which the conjugacy classes of non-central elements are all of the same size n (say), then G must be nilpotent and $n = p^m$ for some prime integer p and some positive integer m. In 2002, K. Ishikawa [78] proved that the nilpotency class of these groups can not be more than 3. In 2006, A. Mann [93] proved that for any finite nilpotent group G, $M(G)$ is nilpotent of class at most 3. Recently M. Isaacs [76, Theorem A] generalized these results for non-nilpotent groups. He proved that if a finite group G contains a normal abelian subgroup A such that $C_G(A) = A$, then $M(G)$ is nilpotent, and it has nilpotency class at most 3. Isaacs' result shows that if G is a finite group which is either nilpotent or supersolvable, then $M(G)$ is nilpotent of class at most 3. In continuation, A. Mann [94] proved that $M(G)$ is nilpotent of class at most 3 if either $M(G)$ is solvable and contains a normal subgroup N with abelian Sylow subgroups such that G/N is nilpotent, or G is solvable and contains a normal subgroup N with abelian Sylow subgroups such that G/N is supersolvable.

In the following theorem, we generalize (at least in principle) Isaacs' result [76, Theorem A] mentioned above.

Theorem 4.11. *Let G be a finite group that contains a normal subgroup A such that $C_G(A) \leq A$ and $[A, x]$ is a normal subset of A for all small elements $x \in G$. Then $M(G)$ is nilpotent, and it has nilpotency class at most 3.*

A subset S of a group G is said to be *normal* if $g^{-1}Sg = S$ (or equivalently $g^{-1}Sg \subseteq S$) for all $g \in G$. Notice that $[A, x]$ is a normal subset of A if it is contained in $Z(A)$. Thus Theorem A gives the following result, which in turn readily gives Isaacs' result.

Corollary 4.12. *Let G be a finite group that contains a normal subgroup A such that $C_G(A) \leq A$ and $[A, x]$ is contained in the center of A for every small element x of G. Then $M(G)$ is nilpotent of class at most 3.*

5. Schur Multiplier

This section consists of the results from the following two papers:

- *The Schur multiplier of central product of groups*, J. Pure Appl. Alg. **222** (2018), 3293-3302. (jointly with Sumana Hatui and L. R. Vermani)
- *The Schur multiplier of groups of order p^5*, J. Group Theory, Ahead of Print (2019), 41 pages. (jointly with Sumana Hatui and Vipul Kakkar)

The Schur multiplier of a given group G, introduced by Schur in 1904 [108], is the second cohomology group $\mathrm{H}^2(G, \mathbb{C}^\times)$ of G with coefficients in $\mathbb{C}^\times$ (upto dual) which is denoted by $\mathrm{M}(G)$. Let G be the direct product of two groups H and K. Then the formulation of the Schur multiplier of G in terms of the Schur multipliers of H and K was given by Schur himself [109]. Such a formulation, when G is a semidirect product of groups H and K, was given by Tahara [112].

We say that G is internal central product of its two normal subgroups H and K amalgamating A if $G = HK$ with $A = H \cap K \subseteq Z(G)$, the center of G and $[H, K] = 1$. Let G be a finite group which is a central product of its subgroups H and K amalgamating A. Recall that $\mathrm{H}^2(G, D)$ denotes the second cohomology group of a group G with coefficients in D, where D is a divisible abelian group regarded as a trivial G-module. For this section, we assume that G is a central product of its normal subgroups H and K with $A = H \cap K$. Set $Z = H' \cap K'$, where X', for a group X, denotes its commutator subgroup. Our aim is to provide upper and lower bounds on $|\,\mathrm{M}(G)|$ in terms of the Schur multipliers of certain quotients of H and K. The following result provides a reduction to the case when $Z = 1$.

Theorem 5.1. *Let B be a subgroup of G such that $B \leq Z$. Then*

$$\mathrm{H}^2(G, D) \cong \mathrm{H}^2(G/B, D)/N,$$

where $N \cong \mathrm{Hom}(B, D)$.

By the tensor product $G_1 \otimes G_2$ of two groups G_1 and G_2, we always mean the abelian tensor product, i.e., $G_1/G_1' \otimes G_2/G_2'$. The following result, along with the preceding result, provides the desired bounds.

Theorem 5.2. *Let $L \cong \mathrm{Hom}\big((A \cap H')/Z, D\big)$, $M \cong \mathrm{Hom}\big((A \cap K')/Z, D\big)$ and $N \cong \mathrm{Hom}(Z, D)$. Then the following statements hold true:*

(i) $\big(\mathrm{H}^2(H/A, D)/L \oplus \mathrm{H}^2(K/A, D)/M\big)/N \oplus \mathrm{Hom}(H/A \otimes K/A, D)$ *embeds in* $\mathrm{H}^2(G, D)$*;*

(ii) $\mathrm{H}^2(G, D)$ *embeds in* $\big(\mathrm{H}^2(H/Z, D) \oplus \mathrm{H}^2(K/Z, D)\big)/N \oplus \mathrm{Hom}(H \otimes K, D)$.

In particular, for $D = \mathbb{C}^*$, assertion (i) provides a generalization of [113] and also of [61] for central product.

It has always been challenging to compute the Schur multiplier of an arbitrary finite p-group with a given presentation. Explicit computations of the Schur multipliers of certain classes of groups not only have applications in mathematics but also in physics. Some efforts have been made to explicitly computing the Schur multiplier for some special classes of groups, e.g., extraspecial p-groups [43], metacyclic group [40, 41], groups of order p^4 [101], groups of order p^2q and p^2qr [80]. Another related topic of current interest is the computation of non-abelian tensor (and exterior) squares of groups, which is extremely useful for computing Schur multipliers. Many mathematicians have studied this topic in the recent past and explicit computations of non-abelian tensor squares of various classes of finite groups have been taken up. For a detailed survey on this topic see [44].

It seems that explicit computations for more classes of finite p-groups will help in understanding the Schur multiplier. With this viewpoint, in this paper, we compute the structure of the Schur multiplier of groups of order p^5. On the way, we also compute non-abelian tensor square and non-abelian exterior square of these groups. As an application we categorise capable and non-capable groups of order p^5. A group G is said to be *capable* if there exists a group H such that $G \cong H/\mathrm{Z}(H)$, where $\mathrm{Z}(H)$ denotes the center of H. This work is done by explicitly considering finite groups of order p^5 from the paper of James [83]. The result is presented in Appendix A in the form of a table.

6. Galois conjugation on characters

This section includes results from

- *Finite groups whose non-linear irreducible characters of the same degree are Galois conjugate,* J. Algebra **452** (2016), 1-16 (jointly with Silvio Dolfi).

In 1992, Berkovich, Chillag and Herzog [39] classified the finite groups whose non-linear irreducible characters all have distinct degrees. Since Galois groups of suitable cyclotomic fields act in a natural degree-preserving way on the set $\mathrm{Irr}\, G$ of a finite group G (see below), it seems natural to weaken the above mentioned condition by asking that there exists just one orbit on $\mathrm{Irr}\, G$ of the irreducible characters for every given irreducible character degree $\neq 1$. While the condition in [39] forces all non-linear characters in $\mathrm{Irr}\, G$ to be rational valued, we are now just imposing a minimality condition on the multiplicities of the degrees of the irreducible characters, without setting restrictions on their fields of values.

Let G be a finite group, n a multiple of $|G|$ and let $\mathcal{G}_n = \mathrm{Gal}(\mathbb{Q}_n|\mathbb{Q})$ be the Galois group of the n-th cyclotomic extension. Then $\mathcal{G}_n$ acts on the set $\mathrm{Irr}\, G$ as follows: for $\alpha \in \mathcal{G}_n$, $\chi \in \mathrm{Irr}\, G$ and $g \in G$, we define

$$\chi^{\alpha}(g) = \chi(g)^{\alpha} \ .$$

For $\chi, \psi \in \mathrm{Irr}\, G$, if there exists a Galois automorphism $\alpha \in \mathcal{G}_n$ such that $\chi^{\alpha} = \psi$, then we say that χ and ψ are *Galois conjugate* (in $\mathcal{G}_n$). This is clearly an equivalence relation on $\mathrm{Irr}\, G$. Characters in the same equivalence class have the same kernel, center, field of values and degree.

In this paper, we weaken the condition of [39], and prove the following result.

Theorem 6.1. *Let G be a finite group. Every two non-linear irreducible characters of the same degree of G are Galois conjugate if and only if G is either abelian or one of the following.*

(a): *G is a p-group (p a prime), $|G'| = p$ and $Z(G)$ is cyclic;*

(b): *G is a Frobenius group with prime power order kernel K and complement L, with L cyclic or $L \cong Q_8$. Moreover:*

- **(b1):** *$L \cong Q_8$ and $|K| = 3^2$; or*
- **(b2):** *K is elementary abelian, $|K| = q^n$ (q prime), L is cyclic and $|L| = (q^n - 1)/d$, where d divides $q - 1$ and $(d, n) = 1$; or*
- **(b3):** *K is a Suzuki 2-group with $|K| = |K'|^2$ and L is cyclic of order $|K'| - 1$.*

(c): *G is non-solvable and either*

$$G \in \{A_5, Sz(8), J_2, J_3, L_3(2), M_{22}, Ru, Th, {}^3D_4(2)\}$$

or

$$G \in \{A_5 \times Sz(8), A_5 \times Th, L_3(2) \times Sz(8)\}.$$

7. Probabilistic Group Theory

This section includes results from the following articles:

- On the probability distribution associated to commutator word map in finite groups, Internat. J. Algebra Comp. **25** (2015), 1107-1124 (jointly with Rajat K. Nath)
- Some bounds on commutativity degree, Rendiconti del Circolo Matematico di Palermo (2) **64** (2015), 229-239 (jointly with Rajat K. Nath)

For a finite group G, let $\mathrm{Pr}(G)$ denote the commutativity degree or commuting probability of G, which is defined by

$$\mathrm{Pr}(G) = |\{(x, y) \in G \times G \mid xy = yx\}|/|G|^2,$$

where, for any finite set S, $|S|$ denotes its cardinality. Various aspects of this notion have been studied by many mathematicians over the years. A very impressive historical account can be found in [73, Introduction]. Pournaki and Sobhani [103] introduced the notion of $\mathrm{Pr}_g(G)$, which is defined by

$$\mathrm{Pr}_g(G) = |\{(x, y) \in G \times G \mid x^{-1}y^{-1}xy = g\}|/|G|^2.$$

Notice that for a given element $g \in G$, $\mathrm{Pr}_g(G)$ measures the probability that the commutator of two randomly chosen group elements is equal to g. Obviously, when $g = 1$, $\mathrm{Pr}_g(G) = \mathrm{Pr}(G)$. Pournaki and Sobhani [103] mainly studied $\mathrm{Pr}_g(G)$ for finite groups G which have only two different irreducible complex character degrees. For such finite groups G which have only two different irreducible complex character degrees 1 and m (say), they proved [103, Theorem 2.2] that for each $1 \neq g \in K(G)$,

$$\mathrm{Pr}_g(G) = (1/|\gamma_2(G)|)(1 - 1/m^2), \tag{7.1}$$

where $K(G)$ denotes the set of all commutators in G. We assume that $1 \in K(G)$.

Motivated by the results of Rusin [105] and the above results, we investigate $\mathrm{Pr}_g(G)$ for some classes of finite groups G which have only two different conjugacy class sizes. It was proved by Dark and Scoppola [53] that the nilpotency class of a finite Camina p-group is at most 3. For such groups

G of nilpotency class 2, we obtain a very nice formula for $\Pr_g(G)$ in the following theorem. Explicit formulas for such groups of class 3 are also obtained.

Theorem 7.2. *Let G be a finite Camina p-group of nilpotency class 2 such that $|\gamma_2(G)| = p^r$, where p is a prime number and r is a positive integer. Then, for $g \in K(G)$,*

$$\Pr_g(G) = \begin{cases} \frac{1}{p^r}\left(1 + \frac{p^r-1}{p^{2m}}\right) & \text{if } g = 1 \\ \frac{1}{p^r}\left(1 - \frac{1}{p^{2m}}\right) & \text{if } g \neq 1, \end{cases}$$

for some positive integer m such that $r < m$.

Denote by $P(G)$ the set $\{\Pr_g(G) \mid 1 \neq g \in K(G)\}$. It follows from [103, Theorem 2.2] that $P(G)$ is a singleton set for all finite groups G which have only two different irreducible complex character degrees. It follows from Theorem A that $P(G)$ is a singleton set for all finite Camina p-groups G of nilpotency class 2. Notice that finite Camina p-groups of class 2 forms a subclass of finite p-groups having only two different conjugacy class sizes. Are there also other classes of finite p-groups G of class 2 having only two different conjugacy class sizes and $|P(G)| = 1$? The answer is affirmative as we show in the following theorem.

Theorem 7.3. *For any positive integer $r \geq 1$, there exists a group G of order $p^{(r+1)(r+2)/2}$ such that $|P(G)| = 1$. Moreover, $\Pr_g(G) = (p^2-1)/p^{2r+1}$ for each $1 \neq g \in K(G)$, and if $r > 1$, G is not a Camina group.*

In the second article we study relative commutativity degree of a subgroup H of a finite group G, denoted by $\Pr(H, G)$, which is the probability that an element of G commutes with an element of H. We obtain some lower and upper bounds for $\Pr(H, G)$ and their consequences. We also study an invariance property of $\Pr(H, G)$ and its generalizations, under isoclinism of pairs of groups.

8. Proposal for future research work

Study of the Schur multiplier interests me a lot. So we are looking forward to exploring certain aspects of the Schur multiplier of finite as well as infinite groups. We hope to explore the surjectivity of the commutator map for finite p-groups. We are also very much interested in the study of the theory left braces and skew left braces in the context of set theoretic solutions of the quantum Yang-Baxter equation.

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Appendix A

G	G^{ab}	$\Gamma(G^{ab})$	$\mathrm{M}(G)$	$G\wedge G$	$G\otimes G$
$\Phi_2(311)a$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p\times\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(8)}$
$\Phi_2(221)a$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)}\times\mathbb{Z}_p^{(7)}$
$\Phi_2(221)b$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p\times\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(8)}$
$\Phi_2(2111)a$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$
$\Phi_2(2111)b$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$
$\Phi_2(2111)c$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(10)}$
$\Phi_2(2111)d$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(10)}$
$\Phi_2(1^5)$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(7)}$	$\mathbb{Z}_p^{(8)}$	$\mathbb{Z}_p^{(18)}$
$\Phi_2(41)$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_p^{(2)}$	$\{1\}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_p^{(3)}$
$\Phi_2(32)a_1$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(4)}$
$\Phi_2(32)a_2$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$
$\Phi_2(311)b$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p\times\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(8)}$
$\Phi_2(311)c$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p\times\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^3}\times\mathbb{Z}_p^{(5)}$
$\Phi_2(221)c$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)}\times\mathbb{Z}_p^{(7)}$
$\Phi_2(221)d$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(4)}\times\mathbb{Z}_p^{(2)}$
$\Phi_3(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_3(2111)b_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_3(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(12)}$
$\Phi_3(311)a$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$
$\Phi_3(311)b_r$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$
$\Phi_3(221)a$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$
$\Phi_3(221)b_r$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}^{(2)}\times\mathbb{Z}_p^{(4)}$
$\Phi_3(2111)c$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_3(2111)d$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(6)}$
$\Phi_3(2111)e$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(6)}$
$\Phi_4(221)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(221)b$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p\times\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(8)}$
$\Phi_4(221)c$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(221)d_r,\ r\neq\frac{1}{2}(p-1)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(221)d_{\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(8)}$
$\Phi_4(221)e$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(221)f_0$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(8)}$
$\Phi_4(221)f_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(9)}$
$\Phi_4(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_4(2111)b$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_4(2111)c$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_4(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(8)}$	$\mathbb{Z}_p^{(14)}$
$\Phi_5(2111)$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$
$\Phi_5(1^5)$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(10)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(16)}$
$\Phi_6(221)a$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$
$\Phi_6(221)b_r,\ r\neq\frac{1}{2}(p-1)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$
$\Phi_6(221)b_{\frac{1}{2}(p-1)}$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$
$\Phi_6(221)c_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$
$\Phi_6(221)d_0$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_{p^2}\times\mathbb{Z}_p^{(5)}$
$\Phi_6(221)d_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\{1\}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$
$\Phi_6(2111)a$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_6(2111)b_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_6(1^5)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(9)}$

G	G^{ab}	$\Gamma(G^{ab})$	$\mathrm{M}(G)$	$G \wedge G$	$G \otimes G$
$\Phi_7(2111)a$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_7(2111)b_r$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_7(2111)c$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(5)}$	$\mathbb{Z}_p^{(11)}$
$\Phi_7(1^5)$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(6)}$	$\mathbb{Z}_p^{(12)}$
$\Phi_8(32)$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p$	$\mathbb{Z}_{p^2} \times \mathbb{Z}_p^{(2)}$	$\{1\}$	$\mathbb{Z}_{p^2}$	$\mathbb{Z}_{p^2}^{(2)} \times \mathbb{Z}_p^{(2)}$
$\Phi_9(2111)a$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_9(2111)b_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_9(1^5)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(6)}$
$\Phi_{10}(2111)a_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_{10}(2111)b_r$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p$	$\mathbb{Z}_p^{(4)}$	$\mathbb{Z}_p^{(7)}$
$\Phi_{10}(1^5)$	$\mathbb{Z}_p^{(2)}$	$\mathbb{Z}_p^{(3)}$	$\mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(3)}$	$\Phi_2(111) \times \mathbb{Z}_p^{(6)}$

TABLE 1. Groups of order p^5, $p \geq 5$