Simple Method of Obtaining Bell Inequalities from Quantum Advantage in One Way-Communication Complexity

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Clauser-Horne-Shimony-Holt inequality

Two distant parties – Alice and Bob – share a quantum state.

Alice measures observable A_x and obtains a result a.

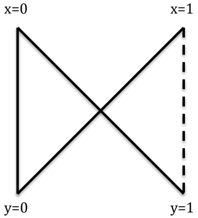
Bob measures observable B_y and obtains a result b.

$$p_{CHSH} = \frac{1}{4}\sum_{x,y=0}^{1} p(a \oplus b = xy|xy)$$

Clauser-Horne-Shimony-Holt inequality

$$p_{CHSH} = \frac{1}{4}\sum_{x,y=0}^{1} p(a \oplus b = xy|xy)$$

$$p_{CHSH} = \frac{1}{4}(p(a \oplus b = 0|0,0) + p(a \oplus b = 0|0,1) + p(a \oplus b = 0|1,0) + p(a \oplus b = 1|1,1))$$



Hence, classical theory can give value

$$p_{CHSH}^{LR} \leq \frac{3}{4}.$$

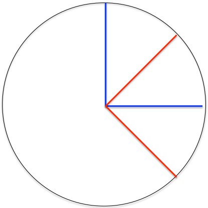
What value of p_{CHSH} can quantum mechanics give?

$$p_{CHSH} = \frac{1}{4}\sum_{x,y=0}^{1} p(a \oplus b = xy|xy)$$

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

Alice performs measurement in one of two blue bases.

Bob performs measurement in one of two red bases.



$$p^Q_{CHSH} \leq \frac{2+\sqrt{2}}{4} = 0.85$$

Scenario

Two distant parties – Alice and Bob

Alice obtains x

Bob obtains y

Task: Bob has to calculate value of a function $f(x, y)$ under the condition that Alice can send to Bob at most k bits of information.

Simple example

Alice obtains two bits x_0 i x_1

Bob obtains bit y

Task: Bob has to give value of a bit x_y, under the condition that Alice can send to Bob at most 1 bit of information.

M. Pawłowski, M. Żukowski, Phys. Rev. A 2010

Simple example – classical strategy

Alice can send to Bob x_0 or x_1.

Bob returns bit sent by Alice.

The probability that bit sent by Alice has value x_y is

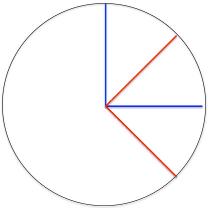
$$p = 0.5 \times 1 + 0.5 \times 0.5 = 0.75$$

Simple example – quantum strategy

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

Alice performs measurement in a basis

$$|a=0\rangle_x = \cos(\tfrac{\pi x}{4})|0\rangle + \sin(\tfrac{\pi x}{4})|1\rangle$$
$$|a=1\rangle_x = \sin(\tfrac{\pi x}{4})|0\rangle - \cos(\tfrac{\pi x}{4})|1\rangle$$
$$\text{where } x = x_0 \oplus x_1 \oplus 1$$

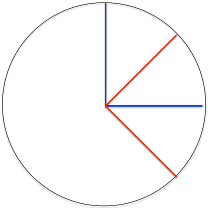


Simple example – quantum strategy

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$$

Bob performs measurement in a basis

$$|b=0\rangle_y = \cos(\tfrac{\pi y}{4} + \tfrac{\pi}{8})|0\rangle + \sin(\tfrac{\pi y}{4} + \tfrac{\pi}{8}))|1\rangle$$
$$|b=1\rangle_y = \sin(\tfrac{\pi y}{4} + \tfrac{\pi}{8})|0\rangle - \cos(\tfrac{\pi y}{4} + \tfrac{\pi}{8}))|1\rangle$$



Simple example – quantum strategy

Alice sends to Bob $m = a \oplus x_0$.

Bob calculates $x_y = m \oplus b$.

$p = 0.85$

Simple example

For optimal classical strategy the probability that Bob gives correct value is 0.75.

For optimal quantum strategy the probability that Bob gives correct value is 0.85.

Universal scheme

Definitions

$$p_A = \sum_{x,y} p(a = 1|x, y)\mu(x, y)$$

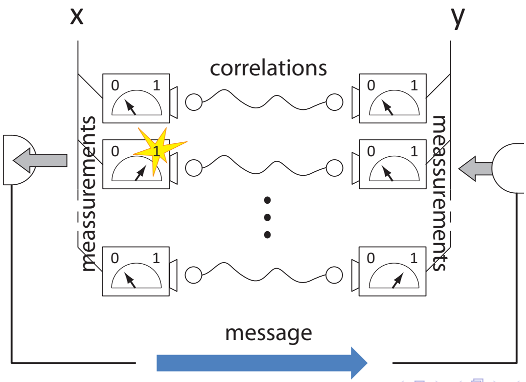
$$p_B = \sum_{x,y} p(b = f(x, y)|x, y, a = 1)\mu(x, y)$$

Ł. Czekaj, A. Grudka, M. Horodecki, P. Horodecki, and M. Markiewicz, arXiv 2013

Universal scheme

Alice performs the same measurement on many copies of state ρ_{AB}.

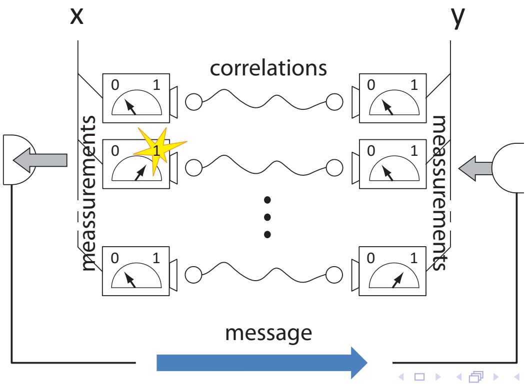
On average one per $m = \frac{1}{p_A}$ measurements will give the correct result.



Universal scheme

Alice has to send to Bob $\log m = -\log p_A$ bits of information in order to identify the correct state.

If Bob performs measurement on this state then with probability p_B he calculates a value of $f(x, y)$.



Universal scheme

Let us suppose that in order to calculate a value of $f(x, y)$ with probability p_B using classical resources Alice has to send to Bob $C(p_B, n)$ bits of classical information.

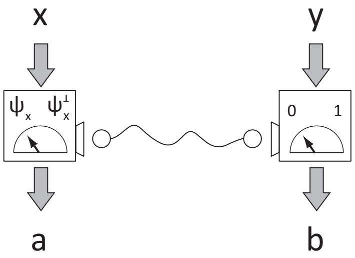
Bell inequality

$$-\log p_A \geq C(p_B, n)$$

Remote state preparation

Alice and Bob share a maximally entangled state of dimension d.

Alice performs measurement in a basis containing vector $|\psi_x\rangle^*$.



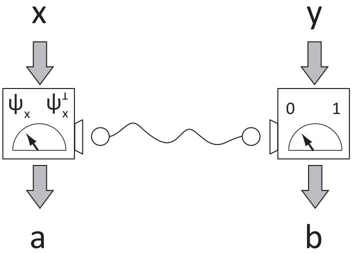
A. Pati, Phys. Rev. A 2000

C. H. Bennett et al, Phys. Rev. Lett. 2001

Remote state preparation

If Alice obtains a result $|\psi_x\rangle^*$ then she prepares on Bob's side a state $|\psi_x\rangle^*$.

The result $|\psi_x\rangle^*$ happens with probability $p_A = \frac{1}{d}$.



A. Pati, Phys. Rev. A 2000

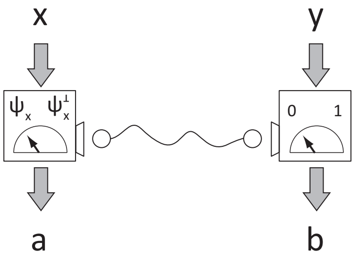
C. H. Bennett et al, Phys. Rev. Lett. 2001

Universal scheme

There exist quantum protocols which require sending $Q(p_B, n)$ qubits in a joint state $|\psi_x\rangle$.

Alice and Bob share many copies of maximally entangled state of dimension $d = 2^Q$.

Alice performs on each copy measurement in a basis containing vector $|\psi_x\rangle^*$.

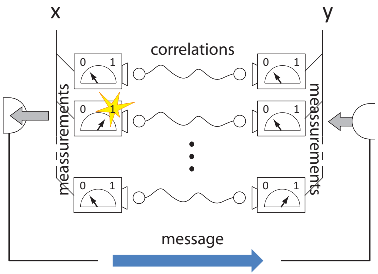


Universal scheme

On average one per $m = \frac{1}{p_A}$ measurements she obtains a result $|\psi_x\rangle^*$ and prepares on Bob's side a state $|\psi_x\rangle^*$.

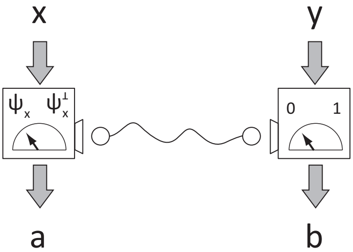
Alice has to send to Bob $\log d = Q(p_B, n)$ bits of information, in order to identify the correct state.

It is like Alice would send $Q(p_B, n)$ qubits.



Universal scheme

If Bob performs measurement on this state then with probability p_B he calculates a value of $f(x, y)$.



Universal scheme

Alice and Bob share $\frac{k}{p_A}$ copies of state

Probability that Alice obtains the correct result for at least one measurement is

$$p'_A = 1 - (1 - p_A)^{\frac{k}{p_A}}$$

$$p'_A \geq 1 - 2^{-k}$$

Probability of success

$$p_S = (1 - \delta)p_B + \frac{\delta}{2}$$

$$\delta = 2^{-k}$$

Universal scheme

The number of bits that Alice has to send to Bob

$$\log \frac{k}{p_A} + 1 = \log \frac{1}{p_A} + \log \log \frac{1}{\delta} + 1$$

Bell inequality

$$\log \frac{1}{p_A} + \log \log \frac{1}{\delta} + 1 \geq C((1-\delta)p_B + \frac{\delta}{2}, n)$$

Universal scheme

$$C(\tfrac{2}{3}, n)$$

$$C(p, n) \geq \tfrac{1}{3}(p - \tfrac{1}{2})^2 C(\tfrac{2}{3}, n)$$

$$\text{for } \tfrac{1}{2} < p < \tfrac{2}{3}$$

Otherwise repeating the protocol several times and taking majority voting the parties can achieve smaller $C(\frac{2}{3}, n)$.

Example

Vector subspace problem

Alice obtains description of n-dimensional vector v.

Bob obtains description of two $\frac{n}{2}$-dimensional subspaces H i $H^{\perp}$.

Assumption: $v \in H$ or $v \in H^{\perp}$.

Problem: Bob has to decide if $v \in H$, or if $v \in H^{\perp}$.

Example

Classical strategy

$$C(2/3, n) \geq cn^{1/3}$$

Quantum strategy

$$\Phi_{\text{iso}} = p\Phi_n^+ + (1-p)\frac{\mathbb{I}}{n^2}$$

$$p_A = \frac{1}{n}$$

$$p_S \geq \frac{2}{3}$$

$$\log n - \log\log\delta > cn^{1/3}$$

Example

Classical strategy

$$C(2/3, n) \geq cn^{1/3}$$

Quantum strategy

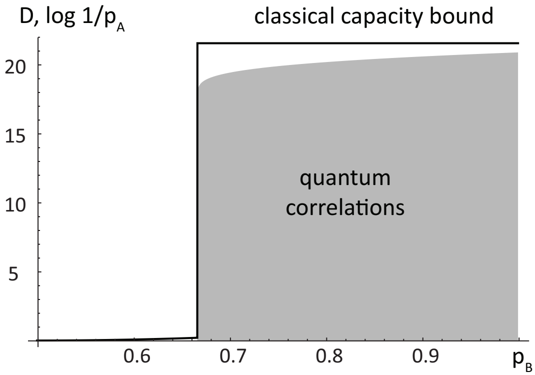
$$\Phi_{\text{iso}} = p\Phi_n^+ + (1-p)\frac{\mathbb{I}}{n^2}$$

$$p_A = \frac{1}{n}$$

$$p_S < \frac{2}{3}$$

$$\log n - \log\log\delta > c\frac{1}{3}(p - \frac{1}{2})^2 n^{1/3}$$

Example



D, log 1/p_A
classical capacity bound
20
15
10
5
quantum
correlations
0.6
0.7
0.8
0.9
p_B

Conclusions

Universal scheme for obtaining Bell inequalities from quantum advantage in communication complexity.

Violation of constructed Bell inequality on example of vector subspace problem.