J. Analysis
Volume 17 (2009), 47–59

Schrödinger Equation, a Survey on Regularity Questions

P.K. Ratnakumar

Abstract. We discuss the regularity questions for the Schrödinger propagator e^{-itL} for the Hermite and special Hermite operator. Essentially the Strichartz estimates and the analyticity properties of the Schrödinger propagator are discussed. The result for the case of Hermite operator is a minor improvement than the one obtained in [11],[12].

Keywords. Schrödinger propagator, Hermite operator, special Hermite operator, Regularity.

2000 MSC. Primary 33C 45, Secondary 35G10.

1. Introduction

The free Schrödinger equation on $\mathbb{R}^n$ is the partial differential equation given by

$$i\partial_t u(x,t) + \Delta u(x,t) = 0, \; x \in \mathbb{R}^n, \; t \in \mathbb{R}, \tag{1.1}$$

where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$ is the laplace operator on $\mathbb{R}^n$. In quantum mechanics, the solution $u(x,t)$ is known as wave function and $|u(x,t)|^2$ gives the probability density of finding a free particle in a given region at a given time. Knowing the initial data,

$$u(x,0) = f(x) \tag{1.2}$$

one would like to know the evolution of this quantum system, which amounts to solving the above PDE with the given initial data f. When $f \in L^2(\mathbb{R}^n)$, this equation can be solved, in a fairly elementary way, by Fourier method.

In fact taking the Fourier transform in the x variable, the above equation, reduces to the following initial value problem for the ordinary differential equation

$$\begin{aligned} i\partial_t \hat{u}(\xi,t) &- |\xi|^2\hat{u}(\xi,t) = 0, \; \xi \in \mathbb{R}^n, \; t \in \mathbb{R}. \\ \hat{u}(\xi,0) &= \hat{f}(\xi). \end{aligned}$$

ISSN 0971-3611

Solving this initial value problem for the ODE in t, treating ξ as a parameter, we see that $\hat{u}(\xi,t) = Ce^{-it|\xi|^2}$ and from the initial data for the ODE, we conclude that the constant $C = \hat{f}(\xi)$, leading to $\hat{u}(\xi,t) = \hat{f}(\xi)\, e^{-it|\xi|^2}$. Taking the inverse Fourier transform, the solution is given by

$$u(x,t) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{-it|\xi|^2} e^{ix\xi} d\xi.$$

Notice that by Fourier inversion formula, f has the representation

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi)\, e^{ix\xi} d\xi.$$

Thus the solution operator $f \to u(x,t)$ is a Fourier multiplier operator, with the multiplier given by the oscillatory function $m(\xi) = e^{-it|\xi|^2}$, reflecting the wave nature of the solution $u(x,t)$. Symbolically the solution is written as $u(x,t) = e^{it\Delta} f(x)$. Thus we see that, the solution is given by the one parameter "oscillatory group" $\{e^{it\Delta} : t \in \mathbb{R}\}$, generated by the operator $i\Delta$. In fact, the operator $e^{it\Delta}$ has the following integral representation as a convolution operator

$$e^{it\Delta} f(x) = \frac{c_n}{t^{n/2}} \int_{\mathbb{R}^n} f(y) e^{\frac{-i|x-y|^2}{4t^2}}\, dy. \tag{1.3}$$

More generally one can consider the oscillatory one parameter group e^{-itL} associated to a self adjoint differential operator L, and in that case the the function $u(x,t) = e^{-itL} f(x)$ (defined using the spectral theory for L), solves the initial value problem for the Schrödinger equation for L:

$$i\partial_t u(x,t) = Lu(x,t),\ x \in \mathbb{R}^n,\ t \in \mathbb{R}.$$
$$u(\cdot,0) = f \in L^2(\mathbb{R}^n).$$

Of particular interest is the case when $L = -\Delta + V$, which corresponds to the motion of a quantum particle in a potential field V.

Since the Schrödinger propagator $e^{it\Delta}$ is given by a Fourier multiplier operator, a natural tool to study the free Schrödinger equation is the Fourier analysis. But Fourier analysis is essentially the spectral theory of the Euclidean Laplacian. From this point of view, one can propose an analogous study of Schrödinger equation for L, using the spectral theory of L. The aim of this lecture is to illustrate this idea, taking two operators of interest in in harmonic analysis, namely Hermite and Special Hermite operators, whose spectral theory is well known. From the application point of view, they deal with, respectively, a quantum particle in a scalar potential field $V(x) = |x|^2$ on $\mathbb{R}^n$ or a particle in a magnetic vector field $X(x,y) = (-y,x) \in \mathbb{C}^n$, where $\Re(z) = x \in \mathbb{R}^n$.

Here we address a couple of regularity questions for the Schrödinger equations, and do a brief survey on results around some of my work in this direction. The first question is about the regularity in terms of Sobolev spaces, i.e., can $u(x,t)$, for $t>0$, have better regularity than the initial data $f \in L^2(\mathbb{R}^n)$? It is natural to measure the regularity using a scale of Sobolev spaces W_L^s, defined in terms of the operator L:

$$W_L^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : L^s f \in L^2(\mathbb{R}^n)\}$$

where L^s is defined using the spectral theory:

$$L^s f = \sum_{k=0}^{\infty} \lambda_k^s P_k f.$$

where P_k denote the projection onto the eigenspace corresponding to the eigenvalue λ_k (For simplicity, we assume that the spectrum of L is the eigenspectrum).

The answer to this question is easily seen to be in the negative (even if L is a smoothing operator!). Notice that for $s>0$, $W_L^s(\mathbb{R}^n)$ is a proper subspace of $L^2(\mathbb{R}^n)$, for nice L, say for instance L is Hypo elliptic. In fact, for $t \in \mathbb{R}$, the operators e^{-itL} are unitary, in particular surjective, hence cannot map L^2 to into $W_L^s(\mathbb{R}^n)$ for $s>0$. This rules out the possibility of e^{-itL} having any regularity in terms of the Sobolev spaces $W_L^s(\mathbb{C}^n)$. This is a general feature of such oscillatory groups.

However when f has nice decay, there seems to be some smoothing property for the oscillatory semigroup e^{itL}. In fact, A Jenson has proved the following weighted Sobolev estimate:

Theorem 1.4. *(A Jenson 1986) Let $L = -\Delta + V$, If $V \in L_k^\infty \in (\mathbb{R}^n)$, (the space of functions having distribution derivatives up to order k in $L^\infty(\mathbb{R}^n)$) for some $k \geq 0$, then $u(x,t) = e^{itL} f(x)$ satisfies the inequality*

$$\int_{\mathbb{R}^n} |(1+\Delta)^{k/2}[u(x,t)(1+|x|^2)^{-k/2}]|^2 dx \leq C \int_{\mathbb{R}^n} |f(x)|^2 (1+|x|^2)^k dx.$$

The following theorem of Hayashi and Saito, regarding the analyticity property of the solutions of $e^{it\Delta}$ can be thought of as a limiting case of the results of Jenson in the case of dimension one.

Theorem 1.5. *(N. Hayashi, S. Saitoh, 1990) Let $u(x,t)$ be the solution to the initial value problem for the free Schrödinger equation. If $f \in L^2(\mathbb{R}, e^{\frac{x^2}{2}} dx)$, then for $t \neq 0$, the function $e^{-\frac{ix^2}{2t}} u(t,x)$ has an analytic continuation to the entire complex plane, and $e^{-\frac{iz^2}{2t}} u(t,z) \in L^2(\mathbb{C}, e^{-\frac{y^2}{2t^2}} dxdy), z = x+iy$. Moreover, $u(z,t)$ satisfies the identity*

$$\frac{1}{|t|\sqrt{\pi}} \int_{\mathbb{C}} e^{-\frac{y^2}{t^2}} |e^{-\frac{iz^2}{2t}} u(t,z)|^2 dxdy = \int_{\mathbb{R}} e^{x^2} |f(x)|^2 dx$$

for $t \neq 0$.

In particular, if f has exponential decay, then u has analytic extension to the whole of $\mathbb{C}$. They have also proved similar results for Schrödinger equation with potential assuming similar analytic extension property on the potential V. Their proof relies on certain identities valid in dimension one. However, the result can be obtained directly for arbitrary dimension using the formula (1.3).

These results highlight the fact that the general Schrödinger propagator has nice regularizing effect, if we restrict the initial data to a suitable sub space of $L^2(\mathbb{R}^n)$. But we can still ask, is there some kind of regularity if we consider initial data from the whole of $L^2(\mathbb{R}^n)$? An interesting result in this context is given by the following theorem of Strichartz, for the free Schroedinger equation:

Consider the inhomogeneous problem

$$\begin{aligned} i\partial_t u(x,t) + \Delta u(x,t) &= g(x,t),\ x \in \mathbb{R}^n,\ t \in \mathbb{R} \\ u(x,0) &= f(x) \end{aligned}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$.

Theorem 1.6 (Strichartz)**.** *Let* $f \in L^2(\mathbb{R}^n)$, $g \in L^{\frac{2(n+2)}{n+4}}(\mathbb{R}^n \times \mathbb{R})$. *Then* $u(x,t) = e^{-it\Delta} f(x)$ *solves the above inhomogeneous problem and* $u \in L^{\frac{2(n+2)}{n}}(\mathbb{R}^n \times \mathbb{R})$. *Moreover, u satisfies the inequality*

$$\left(\int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |u(x,t)|^{\frac{2(n+2)}{n}} dx dt \right)^{\frac{n}{2(n+2)}} \leq C \left(\|f\|_2 + \|g\|_{\frac{2(n+2)}{n+4}} \right).$$

The above estimate (known as the Strichartz estimate), asserts that the solution lies in a higher order L^p space, namely $p = \frac{2(n+2)}{n}$. Strichartz estimate is an important tool in establishing the existence of solutions to the non linear Schrödinger equation.

The above result has been extended to Schrödinger equations with potential, by Journe, Soffer and Sogge for a wide class of bounded potentials V(i.e., Schrödinger equation for $L = -\Delta + V$ in our notation). In fact in [6], they proved analogous estimates in the case of bounded potentials satisfying certain point wise decay at infinity. There has been considerable interest in such estimate ever since, as a global regularity result for the solution of Schrödinger equation, see [4] and [5], and the references there in, for results for bounded potentials. Also see [7] for some generalizations.

An interesting case of an unbounded potential is the quadratic potential $V(x) = |x|^2$. In this case $L = -\Delta + |x|^2$ is the Hermite operator, usually denoted by H, representing the harmonic oscillator Hamiltonian in quantum mechanics.

The equation in this case can be thought of as Schrödinger equation for the Hermite operator. This is a joint work with A. K. Nandakumaran, and our proof relies on the harmonic analysis of Hermite functions. We will discuss the basic idea involved in this paper without going into much of technicalities. Before that we introduce the special Hermite operator mentioned above.

Note that the quantum harmonic oscillator Hamiltonian $H = -\Delta + |x|^2$, for $x \in \mathbb{R}^n$ has the representation

$$H = \frac{1}{2}\sum_{j=1}^{n}(A_jA_j^* + A_j^*A_j)$$

in terms of the creation operators $A_j = -\frac{d}{dx_j} + x_j$ and the annihilation operators $A_j^* = \frac{d}{dx_j} + x_j,\ j = 1, 2, \ldots, n$. Consider the analogous operator $\mathcal{L}$ on $\mathbb{C}^n$, given by

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{n}(Z_j\overline{Z}_j + \overline{Z}_jZ_j)$$

where $Z_j = \frac{\partial}{\partial z_j} + \frac{1}{2}\bar{z}_j,\ \overline{Z}_j = -\frac{\partial}{\partial \bar{z}_j} + \frac{1}{2}z_j,\ j = 1, 2, \ldots, n$. Here $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ denote the complex derivatives $\frac{\partial}{\partial_{x_j}} \mp i\frac{\partial}{\partial_{y_j}}$ respectively. The operator $\mathcal{L}$ was introduced by R. S. Strichartz [16], and is known as the special Hermite operator. In explicit terms it has the form

$$\mathcal{L} = -\Delta + \frac{1}{4}|z|^2 - i\sum_{1}^{n}\left(x_j\frac{\partial}{\partial y_j} - y_j\frac{\partial}{\partial x_j}\right) \tag{1.7}$$

where Δ is the Laplacian on $\mathbb{C}^n$.

What makes this operator interesting is that, it is associated to certain convolution structure on $\mathbb{C}^n$, by virtue of which, the solutions to the initial value problems for basic linear differential equations like heat, wave and Schrödinger equation for $\mathcal{L}$ can be expressed in terms of this convolution structure on $\mathbb{C}^n$. Moreover, the Schrödinger equation for $\mathcal{L}$ can be expressed in the form of a Schrödinger equation with a magnetic vector potential A and a scalar potential V:

$$\sum_{j=1}^{2n}(i\partial_j + A_j)^2u + Vu = 0$$

with $V \equiv 0$ and $\partial_j = \partial_{x_j},\ A_j = -y_j$ for $1 \le j \le n$, $\partial_j = \partial_{y_j},\ A_j = x_j$ for $n+1 \le j \le 2n$.

There is a far reaching generalization of Strichartz estimates by M. Keel and T. Tao, for a general one parameter family of operators $U(t) : H \to L^2(X, d\mu)$, defined on a Hilbert space H. In fact they prove a Strichartz estimate for a general class of one parameter family of operators satisfying certain growth/decay conditions in t for all $t \in \mathbb{R}$. They show that for $f \in H$, $u(x,t) = U(t)f(x)$ lies in the mixed L^p space $L^q(\mathbb{R} : L^p(X, d\mu))$ over $\mathbb{R} \times X$, for pairs (q,p) satisfying the admissibility conditions $\frac{1}{q} = \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$.

A crucial step in the Strichartz estimate, is the so called dispersive estimate

$$\|e^{it\Delta} f\|_{L^\infty(\mathbb{R}^n,dx)} \leq \frac{C}{t^{\frac{n}{2}}} \|f\|_{L^1(\mathbb{R}^n,dx)}.$$

In the case of free Schrödinger equation this follows straightaway from the formula (1.3).

However such dispersive estimates are not valid for all $t \in \mathbb{R}$, in the case of Schrödinger propagator for the Hermite or special Hermite operators that we considered. The reason is that they have discrete spectrum, in fact the spectrum is a subset of integers. Consequently, the operator e^{-itL} is periodic in t with period 2π, hence a decay estimate of the above form clearly cannot hold in the case of operators L with discrete spectrum, in general. It is not even clear if such a decay estimate is valid near $t = 0$.

In the case of the Hermite operator, we can formally write an integral representation of the form

$$e^{-itH} f(x) = \int_{\mathbb{R}^n} K_t(x,y) f(y) dy.$$

In fact every $f \in L^2(\mathbb{R}^n)$ has the Hermite expansion

$$f(x) = \sum_{\alpha \in \mathbb{Z}^n_+} \langle f, h_\alpha \rangle h_\alpha(x) = \sum_{k=0}^{\infty} P_k f(x)$$

where

$$P_k f(x) = \int_{\mathbb{R}^n} f(y) \Phi_k(x,y)\, dx = \int_{\mathbb{R}^n} f(y) \left[\sum_{|\alpha|=k} h_\alpha(x) h_\alpha(y) \right] dxdy$$

are the Hermite projection operators. Recall that, the n-dimensional Hermite functions are the tensor product of the one dimensional Hermite functions, i.e., for each multi index α and $x \in \mathbb{R}^n$, $h_\alpha(x) = \Pi_{i=1}^n h_{\alpha_i}(x_i)$. Moreover we, have $H h_\alpha = 2(|\alpha| + n) h_\alpha$, i.e. h_α are eigenfunctions of the Hermite operator with eigenvalue $(2|\alpha| + n)$. So we can define

$$e^{-itH} f(x) = \sum_{k=0}^{\infty} e^{-it(2k+n)} P_k f(x)$$

for $f \in L^2(\mathbb{R}^n)$. The formal expression for the kernel is

$$K_t(x,y) = e^{-int} \sum_k \omega^k \Phi_k(x,y),$$

which has a closed form representation (Mehler's formula)

$$e^{-int} \sum_{k=0}^{\infty} \omega^k \Phi_k(x,y) = e^{-int} \pi^{-\frac{n}{2}} (1-\omega^2)^{-\frac{n}{2}} e^{-\frac{1}{2}\frac{1+\omega^2}{1-\omega^2}(|x|^2+|y|^2)+\frac{2\omega}{1-\omega^2}x\cdot y}, \tag{1.8}$$

which is valid only for $|\omega| < 1$, where as in our case $\omega = e^{-2it}$ which is of absolute value 1. The method, we employed to get around this difficulty is a regularization technique, which we will explain. We state our theorem regarding Strichartz's type estimate for the Schrödinger equation for the Hermite operator H.

Consider the initial value problem for the Schrödinger equation for H

$$i\partial_t u(\xi,t) = Hu(\xi,t),\ x \in \mathbb{R}^n,\ t \in \mathbb{R}. \tag{1.9}$$

$$u(x,0) = f(x) \tag{1.10}$$

Since e^{-itH} is periodic in t with period 2π, it is natural to look for estimate in terms of the mixed L^p space over $[-\pi,\pi] \times \mathbb{R}^n$.

Theorem 1.11. *Let $f \in L^2(\mathbb{R}^n)$ and let $u(x,t) = e^{-itH}f(x)$ be the solution of the initial value problem (1.9), (1.10). Then u is periodic in t and $u \in L^q([-\pi,\pi]; L^p(\mathbb{R}^n))$, for all pairs (q,p) such that*

$$2 < q < \infty,\ \frac{1}{q} \geq \frac{n}{2}\Big(\frac{1}{2} - \frac{1}{p}\Big) \quad or \quad 1 \leq q \leq 2,\ 2 \leq p < \Lambda$$

where $\Lambda = \infty$ for $n = 1, 2$ and $\Lambda = \frac{2n}{n-2}$ for $n > 2$. Further u satisfies the inequality

$$\|u\|_{L^q([-\pi,\pi];\, L^p(\mathbb{R}^n))} \leq C_n \|f\|_2 \tag{1.12}$$

for all $f \in L^2(\mathbb{R}^n)$ for the above ranges of p and q.

Proof. The result presented here is a minor improvement than in [11] and [12], following the arguments as in [13]. We briefly sketch the idea of the proof. We embed the original one parameter group $\{e^{-itH} : t \in \mathbb{R}\}$ into a complex semi group $\{e^{-\eta H} : \Re(\eta > 0)\}$. By the above discussion, the operator $e^{-\eta H}$, has an integral representation

$$e^{-\eta H} f(x) = \int_{\mathbb{R}^n} K_\eta(x,y) f(y) dy$$

where $e^{2nr} K_\eta(x,y)$, is given by RHS of (0.5) with $\omega = e^{-2\eta}$. From this follows the required dispersive estimate,

$$\|e^{-\eta H} f\|_{L^\infty} \leq \frac{C}{|\sin t|^{\frac{n}{2}}} \|f\|_1$$

for the complex semi group. Using this basic estimate, one can obtain the Strichartz' type estimate for the complex semigroup

$$\|e^{-\eta H} f\|_{L^q(\mathbb{R};L^p(\mathbb{R}^n))} \leq \frac{C}{t^{n/2}} \|f\|_2$$

with the constant C independent of $\Re(\eta)$, using the standard T^*T method. The next step is to obtain the result for the original semi group by a limiting argument. For this we find a sequence $\eta_n = r_n + it$ such that

$$e^{-\eta_n H} f \to e^{-itH} f, \text{ for } a.e.\ (x,t), \text{ as } r_n \to 0.$$

First we observe that for each fixed t, $e^{-\eta H} f \to e^{-itH} f$ in $L^2(\mathbb{R}^n)$ as $\Re(\eta) = r \to 0$. Now for any sequence $\eta_j = r_j + it$, we have by orthogonality of P_k,

$$\|e^{-\eta_j H} f - e^{-itH} f\|_2^2 = \sum_k \left| e^{-(r_j+it)\lambda_k} - e^{-it\lambda_k} \right|^2 \|P_k f\|_2^2$$

where $\lambda_k = (2k+n) \geq 0$. Since $\left| e^{-(r_j+it)(\lambda_k)} - e^{-it(\lambda_k)} \right| \leq 2$ and tends to zero as $r_j \to 0$, a dominated convergence argument applied to the above sum, shows that the RHS tends to zero as $r_j \to 0$.

Integrating the above equality with respect to $t \in [-\pi, \pi]$, and again a DCT argument, shows that

$$\int_{-\pi}^{\pi} \int_{\mathbb{R}^n} |e^{-(r_j+it)H} f(x) - e^{-itH} f(x)|^2 dx dt \to 0$$

as $r_j \to 0$. In other words $e^{-\eta_j H} f$ converges to $e^{-itH} f$ in $L^2(\mathbb{R}^n \times [-\pi, \pi])$. Thus we can extract a subsequence of this sequence, denoted by $e^{-\eta_n H} f(x)$, for which

$$\lim_{\Re(\eta_n) \to 0} e^{-\eta_n H} f(x) = e^{-itH} f(x) \tag{1.13}$$

for $a.e.$ $(x,t) \in \mathbb{R}^n \times [-\pi, \pi]$.

Now application of Fatou's lemma twice, with respect to the variable x and t separately, after taking suitable powers gives the required norm estimate for e^{-itH}. This completes the proof. ■

We remark that, in contrast to the Keel-Tao estimate, the result is also valid for the pair (p,q) when $1 < q < 2$ and moreover, no admissibility condition is required in this case. We obtain this improvement, by proving a result on the convolution on the circle, instead of using the usual Hardy-Littlewood-Sobolev estimate. Here is that elementary lemma.

Lemma 1.14. *Let T denote the convolution operator on the circle given by*

$$Tf(t) = \int_{\mathbb{S}^1} K(t-s)f(s)ds.$$

Assume that K belongs to the weak L^p space $L^\rho_W(\mathbb{S}^1)$, for some $\rho > 1$. Then the inequality

$$\|Tf\|_q \leq C_K \|f\|_{q'}$$

is valid for $q = 2\rho$ and also for $1 \leq q \leq 2$, where C_K is a constant depending only on K.

Proof. By the generalized Young's inequality, we have (see [3]):

$$\|Tf\|_r \leq C[K]_\rho \|f\|_{q'}$$

where $[K]_\rho$ denoting the weak $L^\rho(\mathbb{S}^1)$ norm of K, and is valid for all r such that $\frac{1}{q'} + \frac{1}{\rho} = 1 + \frac{1}{r},\ \rho > 1, q' > 1$. Setting $r = q$, this reads

$$\|Tf\|_q \leq C[K]_\rho \|f\|_{q'} \tag{1.15}$$

for $q = 2\rho$. Notice that these arguments are valid for $2 < q < \infty$, since $\rho > 1$ and $q' > 1$ by assumption.

Now we observe that the weak L^p spaces L^ρ_W are in L^1_{loc} for $\rho > 1$: In fact, for any compact set Θ, set $g = f\chi_\Theta$. The distribution function of g is given by $\lambda_g(\alpha) = |\{x : |g(x)| > \alpha\}| = |\{x \in \Theta : |f(x)| > \alpha\}| \leq |\Theta|$. Hence $\lambda_g(\alpha)$ is bounded for $\alpha > 0$. Also $\lambda_g(\alpha) \leq \lambda_f(\alpha) \leq \frac{C}{\alpha^\rho}$, since $f \in L^\rho_W$. These two inequalities yield $\lambda_g(\alpha) \leq \frac{C}{1+\alpha^\rho}$. Thus $\int_\Theta |f| = \|g\|_1 = \int_0^\infty \lambda_g(\alpha)d\alpha \leq \int_0^\infty \frac{d\alpha}{1+\alpha^\rho}$. This integral is finite for $\rho > 1$, showing that $f \in L^1_{loc}$.

By the above observation, we have $K \in L^1(\mathbb{S}^1)$. Hence by Minkowski's inequality for integrals,

$$\|Tf\|_q \leq \|K\|_1 \|f\|_q \text{ for } 1 \leq q \leq \infty.$$

Integrating this inequality for $q = \infty$ over $\mathbb{S}^1$ yields

$$\|Tf\|_1 \leq 2\pi \|K\|_1 \|f\|_\infty.$$

Interpolating this with the above L^q estimate for $q = 2$, we get

$$\|Tf\|_q \leq C\|K\|_1 \|f\|_{q'} \text{ for } 1 \leq q \leq 2.$$

This completes the proof. ∎

To illustrate an application of the above Strichartz estimate, we now consider the inhomogeneous problem:

$$\begin{gathered} i\partial_t u(x,t) - Hu(x,t) = g(x,t),\ x \in \mathbb{R}^n,\ t \in \mathbb{R} \\ u(x,0) = f(x). \end{gathered}$$

In this case the solution is given by the Duhamel's formula :

$$u(x,t) = e^{-itH} f(x) - i \int_0^t e^{-i(t-s)H} g(x,s) ds.$$

Using the Strichartz estimate for e^{-itH}, we prove the following

Theorem 1.16. *Let $f \in L^2(\mathbb{R}^n)$ and $g(x,t) \in L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{R}^n))$ then the solution $u(x,t)$ to the above problem lies in $L^q(\mathbb{S}^1;\ L^p(\mathbb{R}^n))$, for all pairs (q,p) such that $2 < q < \infty,\ \frac{1}{q} \geq \frac{n}{2}(\frac{1}{2} - \frac{1}{p})$ or $1 \leq q \leq 2,\ 2 \leq p < \lambda$, where $\lambda = \infty$ for $n = 1, 2$ and $\lambda = \frac{2n}{n-2}$ for $n > 2$. Further $u(x,t)$ satisfies the inequality*

$$\|u(z,t)\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{R}^n))} \leq C_n(\|f\|_2 + \|g\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))})$$

for the above pairs (p,q) with some constant C_n independent of f and g.

Proof. Follows from Duhamel's formula, and the estimates :

$$\|u\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{R}^n))} \leq C_n \|f\|_2$$

$$\left\| \int_0^t e^{-i(t-s)\mathcal{L}} g(x,s) ds \right\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{R}^n))} \leq C \|g\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{R}^n))}$$

which also follows by arguments as in the proof for Theorem 1.11. ■

2. The case of special Hermite operator

Analogous results hold for the special Hermite operator as well. In fact using the spectral theory of the special Hermite operator, one can show that the corresponding complex semi group has the following integral representation as a twisted convolution operator

$$e^{-\eta\mathcal{L}} f(z) = f \times K_\eta(z) \tag{2.1}$$

with kernel

$$K_\eta(z) = (2\pi)^{-n} \sum_{k=0}^{\infty} e^{-\eta(2k+n)} \varphi_k(z),$$

where φ_k are given in terms of the Laguerre polynomials L_k^{n-1}:

$$\varphi_k(z) = L_k^{n-1}(\frac{1}{2}|z|^2) e^{-\frac{1}{4}|z|^2},$$

see [17] for the definition of Laguerre polynomials.

The relevant Strichartz type estimate, in this case is

$$\|e^{-it\mathcal{L}} f\|_{L^q([-\pi,\pi];\, L^p(\mathbb{C}^n))} \leq C_n \|f\|_2 \tag{2.2}$$

valid for all $f \in L^2(\mathbb{C}^n)$, for all pairs (q,p) such that

$$2 < q < \infty,\ \frac{1}{q} \geq n(\frac{1}{2} - \frac{1}{p}) \quad \text{or} \quad 1 \leq q \leq 2,\ 2 \leq p < \frac{2n}{n-1}.$$

For more details we refer the reader to [13].

The above Strichartz estimate leads to an existence theorem for the inhomogeneous problem as before:

$$\begin{aligned} i\partial_t u(z,t) - \mathcal{L}u(z,t) &= g(z,t),\ z \in \mathbb{C}^n,\ t \in \mathbb{R} \\ u(z,0) &= f(z). \end{aligned}$$

Theorem 2.3. *Let $f \in L^2(\mathbb{C}^n)$ and $g(z,t) \in L^{q'}(\mathbb{S}^1;\ L^{p'}(\mathbb{C}^n))$ then the solution $u(z,t)$ to the above problem lies in $L^q(\mathbb{S}^1;\ L^p(\mathbb{C}^n))$, for all pairs (q,p) such that $1 \leq q \leq 2,\ 2 \leq p < \frac{2n}{2n-1}$ or $2 < q < \infty,\ \frac{1}{q} \geq n(\frac{1}{2} - \frac{1}{p})$. Further $u(z,t)$ satisfies the inequality*

$$\|u(z,t)\|_{L^q(\mathbb{S}^1;\, L^p(\mathbb{C}^n))} \leq C_n(\|f\|_2 + \|g\|_{L^{q'}(\mathbb{S}^1;\, L^{p'}(\mathbb{C}^n))})$$

for the above pairs (p,q) with some constant C_n independent of f and g.

Proof. Follows from Duhamel's formula

$$u(z,t) = e^{-it\mathcal{L}} f(z) - i\int_0^t e^{-i(t-s)\mathcal{L}} g(z,s)ds$$

with the similar arguments as before. For more details see [13]. ■

3. Analyticity Property

As mentioned above the Schrödinger propagator e^{-itL} has some interesting regularity property. Here we state one such result for Schrödinger group $e^{it\mathcal{L}}$ associated to the Special Hermite operator $\mathcal{L}$ on $\mathbb{C}^n$, which concerns the analytic extension property of $e^{-it\mathcal{L}}$.

We think of $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ and write $z = (x,y)$, $x, y \in \mathbb{R}^n$. Let

$$\begin{aligned} S_t(x,u) &= e^{-it\mathcal{L}} g(x,u) \\ &:= \sum_{k=0}^{\infty} e^{-it(2k+n)} g \times \varphi_k(x,u). \end{aligned} \tag{3.1}$$

The following result is a part of a joint work with Thangavelu and Sanjay Parui, which concerns the the analytic extension of $S_t(g)$.

Theorem 3.2. *If $g \in L^2(\mathbb{R}^{2n}, e^{(x^2+u^2)}dx\,du)$, then $S_t(x,u) = e^{-itL}g(x,u)$ extends as an entire function on $\mathbb{C}^{2n}$, for $t \neq n\pi,\ n \in \mathbb{Z}$. Moreover, $S_t(z,w)\, e^{-i\frac{\cot(t)}{4}(z^2+w^2)}$ lies in the Bergman space $L^2(\mathbb{C}^{2n},\ e^{-\frac{\csc^2(t)}{4}(y^2+v^2)}dz\,dw)$ and satisfies the identity*

$$\begin{aligned}
&\|S_t(z,w)\, e^{-i\frac{\cot(t)}{4}(z^2+w^2)}\|_{L^2(\mathbb{C}^{2n}, e^{-\frac{\csc^2(t)}{4}(y^2+v^2)}dz\,dw)} \\
&= \left(\frac{\sin(t)}{\sqrt{\pi}}\right)^n \|g(x,y)\|_{L^2(\mathbb{R}^{2n}, e^{2(x^2+u^2)}dx\,du)}.
\end{aligned}$$

This is a crucial step in proving such analytic extension property of Schrödinger equation for the sublaplacian on the Heisenberg group. For similar results for the Hermite operator and the Heisenberg sublaplacian, we refer the reader to [10]. The result presented in [10] for the special Hermite operator is slightly different, but proof follows by similar reasoning.

References

[1] T. Cazenave, *Semi Linear Schrödinger Equations*, Courant Instt. Lec Notes Series, No. 10, AMS, 2003.

[2] G.B. Folland, *Harmonic Analysis in Phase Space*, Annals of math studies 122, Princeton Univ. Press., Princeton NJ, 1989.

[3] G.B. Folland, *Real Analysis, Modern Techniques and Their Applications*, Wiley-Interscience Publ., New York, 1984.

[4] M. Goldberg and W. Schlag, Dispersive estimates for Schrödinger operator in dimension one and three, *Comm. Math. Phys*, **251**(2004), no. 1, 157–178.

[5] J. Ginibre and G. Velo, The glogal Cauchy problem for the nonlinear SchrÖdinger equation revisited, *Ann. Insttt. Poincare*, **2**(1985), no. 4, 309–327.

[6] J.L. Journe, A. Soffer and C.D. Sogge, Decay estimates for Schrödinger operators, *Communications on Pure and Applied mathematics*, **XLIV**(1991), 573–604.

[7] C. Kenig, G. Ponce, and L. Vega, Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations, *Invent. Math.*, **134**(1998), 489–545.

[8] M. Keel and T. Tao, End point Strichartz estimates, *American J. Math.*, **120**(1998), 955–980.

[9] N.N. Lebedev, *Special Functions and Their Applications*, Dover Publ., New York, 1992.

[10] S. Parui, P.K. Ratnakumar and S. Thangavelu, On analyticity of the Schrödinger propagator, Preprint.

[11] A.K. Nandakumaran and P.K. Ratnakumar, Schrödinger equation and the regularity of the oscillatory semi group for the Hermite operator. *J. Funct. Anal*, **224**(2005), 371–385.

[12] A.K. Nandakumaran and P.K. Ratnakumar, Corrigendum, Schrödinger equation and the regularity of the oscillatory semi group for the Hermite operator, *J. Funct. Anal*, **224**(2006), 719–720.

[13] P.K. Ratnakumar, On Schrödinger propogator for the special Hermite Operator, *J. Fourier Anal. Appl.*, **14**(2008), no. 2, 286–300.

[14] E.M. Stein and G. Weiss, *Fourier Analysis on Euclidean Spaces,* Princeton Univ. Press, Princeton, N.J., 1975.

[15] R.S. Strichartz, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Math J.*, **44**(1977), no.3, 705–714.

[16] R.S. Strichartz, Harmonic analysis as spectral theory of Laplacians, *J. Funct. Anal.*, **87**(1989), 51–148.

[17] S. Thangavelu, *Lectures on Hermite and Laguerre expansions,* Mathematical notes, 42, Princeton Univ. press, Princeton, 1993.

P.K. Ratnakumar E-mail: `ratnapk@hri.res.in`

Address:

Department of Mathematics,
Harish-Chandra Research Institute,
Chhatnag Road, Jhusi,
Allahabad – 211 019, India