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Lectures-1+2: Cold Dark Matter Hypothesis in Cosmology

(*) A model of the observed Universe:

Homogeneous and isotropic background with density perturbations.

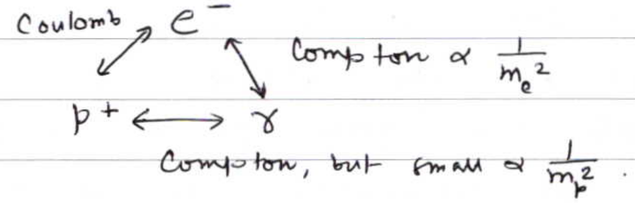
(*) Observational basis: The CMB (redshift $z \simeq 1100$) is isotropic to a high degree of precision: 1 part in 10^3. The small temp. fluctuations $\left(\frac{\Delta T}{T_0}\right)$ of order 10^{-3} have a dipolar pattern (a 'hot' region and a 'cold' region). One part of the dipole is caused by the motion of our solar system w.r.t. to the 'CMB rest frame' — cannot be separated from the cosmological dipole counterpart. ~~Next~~ If we subtract the dipole, $\frac{\Delta T}{T_0} \sim 10^{-5}$. Hence the early Universe is well-described by an FRW + small fluctuations model.

(*) The Problem: The present Universe is very clumpy, with large density fluctuations in various patterns. First (observed) galaxies were formed at least around redshift $z \simeq O(10)$, which require much larger than $O(1)$ density fluctuations. Therefore, how did the ~~structure~~ density fluctuations grow from $O(\text{few}\times 10^{-5})$ at $z \simeq 1100$, to $O(1)$ at least at $z \simeq 10$?

(*) Growth of perturbations due to gravity: Clearly, we need to understand the growth of density perturbations due to gravity. We also need to understand the generation of the perturbations in the first place ("primordial density pert.").

(*) The players: Photons, baryons (proton, neutron, electron, light nuclei etc.), neutrinos — the ones we expect to surely take part.

CMB decoupling temperature ~ 0.26 eV
MR equality " ~ 1 eV
At these temps. protons, electrons, .. are non-relativistic, but photons and neutrinos are relativistic. Until ~~decoupling at recombination at a~~ photon decoupling at $z \sim 1100$, baryons are tightly coupled to photons by Thompson scattering of $e-\gamma$ and Coulomb scattering of $e-p$.



The dynamics, therefore, is fairly complicated. Furthermore, the perturbed gravitational field enters in the phase-space evolution of these energy components, which in turn determine the gravitational field. In general, we need to solve a coupled system of equations for the phase-space distribution of matter and the eqns. for the gravitational field (metric of space-time).

But, part of the essential physics can be captured by studying a gravitating non-relativistic fluid in an expanding Universe, using simple Newtonian dynamics with Newtonian gravity. This approach is sufficient to understand sub-Hubble perturbations in NR matter (Sub-Hubble: perturbations of wave-length $< H^{-1} \sim$ curvature scale of the FRW space-time).

(*) Ultimately, it is the NR matter that will cluster, hence the emphasis on studying them. But there are aspects of rel. matter that influence the details of ~~str~~ structure formation.

Non-relativistic fluid dynamics :

Continuity eqn. (mass conservation in a volume)

$$\left(\frac{\partial \rho}{\partial t}\right)_{\vec{r}} + \underbrace{\vec{\nabla}_{\vec{r}} \cdot (\rho \vec{u})}_{\text{flux}} = 0 \quad -(1)$$

$\vec{u}$: fluid velocity field

$\vec{r}$: physical co-ordinate

Euler eqn.

$$\underbrace{\rho\left(\frac{\partial \vec{u}}{\partial t}\right)_{\vec{r}} + \rho\, \vec{u} \cdot \vec{\nabla}_r \vec{u}}_{= \rho\left(\frac{d\vec{u}}{dt}\right) \text{ (mass x acceleration)}} = \underbrace{- \vec{\nabla}_r P}_{\text{pressure - force}} \underbrace{- \rho \vec{\nabla}_r \Phi}_{\text{gravitational force}} \quad -(2)$$

Poission Eqn. : The gravitational potential Φ satisfies .

$$\nabla_r^2 \Phi = 4\pi G \rho \quad -(3)$$

In an expanding spatially flat FRW Universe, it is useful to introduce "co-moving" co-ordinates, in terms of which the background metric is $ds^2 = dt^2 - a^2(t)\delta_{ij}dx^i dx^j$

Co-ordinates appearing in the fluid eqns. :

$\vec{r}$: physical co-ordinates

$\vec{x}$: co-moving "

$$\vec{r}(t) = a(t)\,\vec{x}$$

velocity field : $\vec{u}(t) = \dot{\vec{r}}(t) = \dot{a}(t)\,\vec{x} + a(t)\,\dot{\vec{x}}$

$$= H(t)\,\vec{r} + a(t)\,\dot{\vec{x}} \equiv H(t)\,\vec{r} + \vec{v}(t)$$

Here, $\vec{v} = a\dot{\vec{x}} \left(= a\frac{d\vec{x}}{dt}\right)$ is the physical peculiar velocity, while $H\vec{r}$ is the Hubble flow.

Now, consider

Ex.1. $$\left(\frac{\partial f(t, \underbrace{a(t)\vec{x}}_{\vec{r}})}{\partial t}\right)_{\vec{x}} = \underbrace{\left(\frac{\partial f}{\partial t}\right)_{\vec{r}}}_{\text{hold } \vec{r} = a(t)\vec{x} \text{ fixed}} + \underbrace{\frac{\partial}{\partial t}\left(a(t)\,\vec{x}\right)}_{\text{Change in } \vec{r} \text{ w.r.t. } t} \cdot \underbrace{\left(\frac{\partial f}{\partial \vec{r}}\right)_t}_{\text{hold } t\text{-fixed}}$$

$$= \left[\left(\frac{\partial}{\partial t}\right)_{\vec{r}} + H\,\vec{r}\cdot\vec{\nabla}_{\vec{r}}\right] f$$

or, $$\left(\frac{\partial f}{\partial t}\right)_{\vec{r}} = \left(\frac{\partial f}{\partial t}\right)_{\vec{x}} - H\,\vec{x}\cdot\vec{\nabla}_x f$$

Now, w/ x-fixed

$$\frac{\partial}{\partial t}\left(a(t)\,\vec{x}\right) = \dot{a}(t)\,\vec{x} = \frac{\dot{a}}{a}\,a\vec{x} = H\vec{r}$$

Therefore, in the co-moving co-ordinates, the continuity eqn. is:

$\vec{\nabla}_{\vec{r}}$: gradient wrt $\vec{r}$ at fixed t

$$\vec{\nabla}_{\vec{r}} = a^{-1}\,\vec{\nabla}_{\vec{x}}$$

$$\Rightarrow \vec{r}\cdot\vec{\nabla}_r = \vec{x}\cdot\vec{\nabla}_{\vec{x}}$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{\vec{r}} + \left(\vec{\nabla}_{\vec{r}}\cdot\rho\vec{u}\right) = 0$$

$$\Rightarrow \left(\left(\frac{\partial}{\partial t}\right)_{\vec{x}} - \underline{H\,\vec{x}\cdot\vec{\nabla}_x}\right)\rho + \frac{1}{a}\,\vec{\nabla}_{\vec{x}}\cdot(\rho\vec{u}) = 0$$

Now, $$\frac{1}{a}\,\vec{\nabla}_{\vec{x}}\cdot(\rho\vec{u}) = \frac{1}{a}\,\vec{\nabla}_{\vec{x}}\cdot\left(\rho H\vec{r} + \rho\vec{v}\right)$$

$$= \frac{1}{a}\cdot aH\,\cancel{\vec{\nabla}_{\vec{x}}}\;\vec{\nabla}_{\vec{x}}\cdot(\rho\vec{x}) + \frac{1}{a}\,\vec{\nabla}_{\vec{x}}\cdot(\rho\vec{v})$$

$$= \underbrace{H\vec{x}\cdot\vec{\nabla}_{\vec{x}}\rho}_{\text{Cancels}} + H\rho\,\underbrace{\left(\vec{\nabla}_{\vec{x}}\cdot\vec{x}\right)}_{=3} + \frac{1}{a}\,\vec{\nabla}_{\vec{x}}\cdot(\rho\vec{v})$$

Hence, $$\boxed{\left(\frac{\partial \rho}{\partial t}\right)_{\vec{x}} + 3H\rho + \frac{1}{a}\,\vec{\nabla}_{\vec{x}}\cdot(\rho\vec{v}) = 0} \quad — (4)$$

Continuity eqn. in an expanding FRW universe

Henceforth, all $\left(\frac{\partial}{\partial t}\right) = \left(\frac{\partial}{\partial t}\right)_{\vec{x}}$ and $\vec{\nabla}_{\vec{x}} \equiv \vec{\nabla}$

Ex. 2. It is straightforward to check that the Euler eqn. in co-moving co-ordinates is

$$\boxed{\left(\frac{\partial \vec{u}}{\partial t}\right)_{\vec{x}} + \frac{\vec{v}}{a} \cdot \vec{\nabla}_x \vec{u} = -\frac{1}{a} \frac{\vec{\nabla}_x P}{\rho} - \frac{1}{a} \vec{\nabla}_x \Phi} \quad - (5)$$

where, a term such as $\vec{u} \cdot \vec{\nabla}_r \vec{u} = u_x(\nabla_x \vec{u}) + u_y(\nabla_y \vec{u}) + u_z(\nabla_z \vec{u})$

Finally, the Poission eqn. is

$$\nabla_r^2 \Phi = 4\pi G \rho$$

$$\Rightarrow \boxed{\nabla_x^2 \Phi = 4\pi G a^2 \rho} \quad - (6)$$

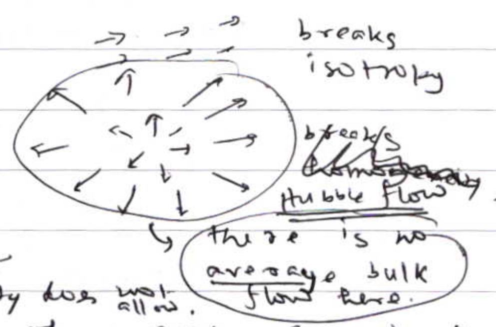
For small perturbations, density $\rho(t, \vec{x}) = \bar{\rho} + \delta\rho(t, \vec{x})$
pressure, $P(t, \vec{x}) = \bar{P} + \delta P(t, \vec{x})$

(*) Since the perturbations are small, we can linearize the fluid eqns.

$$AB = (\bar{A} + \delta A)(\bar{B} + \delta B)$$
$$= \underbrace{\bar{A}\bar{B}}_{\text{zeroth order}} + \underbrace{\bar{A}\,\delta B + \bar{B}\,\delta A}_{\text{1st order}} + \underbrace{\delta A\, \delta B}_{\text{2nd order}}$$

$$\xrightarrow{\text{linearize}} \bar{A}\bar{B} + \bar{A}\,\delta B + \bar{B}\,\delta A$$

The peculiar velocity field $\vec{v} = 0$ in the FRW background solns., as ~~that~~ a non-zero $\vec{v}$ field will break ~~iso~~ homogeneity + isotropy in general.



Background solns: $\vec{v} = 0$, $\rho = \bar{\rho}$, $P = \bar{P}$

Also a bulk flow in a dirn. requires density gradient → homogeneity does not allow.

By isotropy, the mean value of any 3-vector field $v^i = 0$. Hence, in ~~an~~ strictly FRW universe, only Hubble flow takes place, and the peculiar vel. are ~~negligibly small~~ zero.

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\vec{\nabla}\cdot(\rho\vec{u}) = 0$$

at zeroth order this continuity eqn. becomes

$$\frac{\partial \bar{\rho}}{\partial t} + 3H\bar{\rho} = 0 \Rightarrow \bar{\rho} \propto a^{-3}$$ as for a NR fluid in FRW.

The Euler eqn.

$$\frac{\partial \vec{u}}{\partial t} + \frac{\vec{v}}{a}\cdot\vec{\nabla}\vec{u} = -\frac{1}{a}\frac{\vec{\nabla} P}{\rho} - \frac{1}{a}\vec{\nabla}\Phi$$

at zeroth order $\Rightarrow$ $$\frac{\partial \vec{\bar{u}}}{\partial t} = -\frac{1}{a}\frac{\vec{\nabla}\bar{P}}{\bar{\rho}} - \frac{1}{a}\vec{\nabla}\bar{\Phi}$$

now, $\vec{\bar{u}} = H\vec{r} = Ha\vec{x}$

and $\bar{P}$ = constant (isotropic + homogeneous)

$\simeq 0$ for NR matter.

Hence, $$\vec{\nabla}\bar{\Phi} = -a\frac{\partial \vec{\bar{u}}}{\partial t} = -a\ddot{a}\vec{x}$$ $\boxed{\frac{\partial}{\partial t}Ha = \frac{\partial}{\partial t}\dot{a} = \ddot{a}}$

$$\Rightarrow -a\ddot{a}(\vec{\nabla}\cdot\vec{x}) = \nabla^2\bar{\Phi}$$

$$\Rightarrow -a\ddot{a}\underbrace{(\vec{\nabla}\cdot\vec{x})}_{=3} = 4\pi G a^2\bar{\rho}$$ (by the zeroth order Poission eqn.)

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\bar{\rho}$$

[Recall, in FRW geometry, we had the 2nd Freidmann eqn. $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{P})$

$\simeq -\frac{4\pi G}{3}\bar{\rho}$ for NR matter with $\bar{P}\simeq 0$.]

<u>Perturbations</u>:

$$\rho = \bar{\rho}[1+\delta], \quad \vec{u} = \vec{\bar{u}} + \vec{v}$$

$\vec{\bar{u}} = Ha\vec{x}$ (Hubble flow only)

$$P = \bar{P} + \delta P, \quad \Phi = \bar{\Phi} + \delta\Phi$$

Density contrast: $\delta = \frac{\rho - \bar{\rho}}{\bar{\rho}} = \frac{\delta\rho}{\bar{\rho}}$

$$\Rightarrow \delta\rho = \bar{\rho}\,\delta$$

For $\delta \ll 1$, can linearize the eqns.

$\bar{\rho} = \bar{\rho}(t)$ only by homogeneity.

* **Continuity eqn.** : $\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a}\vec{\nabla}\cdot(\rho\vec{v}) = 0$ ↳ 1st order already

1st order: $\frac{\partial}{\partial t}(\bar{\rho}\delta) + 3H(\bar{\rho}\delta) + \frac{1}{a}\vec{\nabla}\cdot(\bar{\rho}\vec{v}) = 0$

$$\Rightarrow \delta\frac{\partial\bar{\rho}}{\partial t} + \bar{\rho}\frac{\partial\delta}{\partial t} + 3H\bar{\rho}\delta + \frac{\bar{\rho}}{a}(\vec{\nabla}\cdot\vec{v}) = 0$$

$$\Rightarrow \delta\underbrace{\left(\frac{\partial\bar{\rho}}{\partial t} + 3H\bar{\rho}\right)}_{=0 \text{ by the zeroth order eqn.}} + \bar{\rho}\frac{\partial\delta}{\partial t} + \frac{\bar{\rho}}{a}(\vec{\nabla}\cdot\vec{v}) = 0$$

$$\Rightarrow \boxed{\frac{\partial\delta}{\partial t} = -\frac{1}{a}\vec{\nabla}\cdot\vec{v}} \quad (*1)$$

growth of density perturbations ↗ ; ↓ sourced by div. of fluid vel.

Change in density contrast is related to the divergence of peculiar flow due to continuity.

* **Euler eqn.** : $\frac{\partial\vec{u}}{\partial t} + \frac{\vec{v}}{a}\cdot\vec{\nabla}\vec{u} = -\frac{1}{a}\frac{\vec{\nabla}P}{\rho} - \frac{1}{a}\vec{\nabla}\Phi$

1st order: $\dot{\vec{v}} + \frac{\vec{v}}{a}\cdot\vec{\nabla}(Ha\vec{x}) = -\frac{1}{a}\frac{\vec{\nabla}\delta P}{\bar{\rho}} - \frac{1}{a}\vec{\nabla}(\delta\Phi)$

~~When $\delta P = 0$, and $\delta\Phi \neq 0$~~

$$\Rightarrow \boxed{\dot{\vec{v}} + \cancel{\frac{\vec{v}}{a}\times Ha}\; H\vec{v} = -\frac{1}{a\bar{\rho}}\vec{\nabla}\delta P - \frac{1}{a}\vec{\nabla}\delta\Phi} \quad (*2)$$

Now if $\delta P = 0$, $\delta\Phi = 0$, then

$$\dot{\vec{v}} + H\vec{v} = 0 \Rightarrow \frac{d}{dt}(a\vec{v}) = 0 \Rightarrow \boxed{\vec{v} \propto a^{-1}}$$

Time derivative of (*1) gives

$$\ddot{\delta} - \frac{1}{a^2}\dot{a}(\vec{\nabla}\cdot\vec{v}) + \frac{1}{a}\vec{\nabla}\cdot\frac{\partial\vec{v}}{\partial t} = 0$$

$$\Rightarrow \ddot{\delta} - \frac{H}{a}(\vec{\nabla}\cdot\vec{v}) + \frac{1}{a}\vec{\nabla}\cdot\left(-H\vec{v} - \frac{1}{a\bar{\rho}}\vec{\nabla}\delta P - \frac{1}{a}\vec{\nabla}\delta\Phi\right) = 0$$

$$\Rightarrow \ddot{\delta} + 2H\dot{\delta} - \frac{1}{a^2}\left(\frac{1}{\bar{\rho}}\nabla^2\delta P + \nabla^2\delta\Phi\right) = 0$$

(as $\dot{\delta} = -\frac{\vec{\nabla}\cdot\vec{v}}{a}$)

~~$\delta\left(\frac{a}{b}\right)$~~

$$\frac{\bar{a} + \delta a}{\bar{b} + \delta b} \quad \left| \begin{array}{l} a = \vec{\nabla}P \\ b = \rho \end{array} \right.$$

$= (\bar{a} + \delta a)(\bar{b} - \delta b)$

$= \bar{a}\bar{b} + \boxed{\bar{b}\delta a - \bar{a}\delta b} + O(\delta^2)$

$-\vec{\nabla}\bar{P}(\bar{\rho}\bar{\delta})$

$= 0$ as $\vec{\nabla}\bar{P} = 0$

by the FRW. $\bar{P}$ = constant

$\vec{v}\cdot\vec{\nabla}\vec{x}$

$= v_x\frac{\partial}{\partial x}\vec{x} + v_y\frac{\partial}{\partial y}\vec{x} + v_z\frac{\partial}{\partial z}\vec{x}$

$= v_x\hat{e}_x + v_y\hat{e}_y + v_z\hat{e}_z$

$= \vec{v}$

Now, consider a fluid for which

$$P = P(\rho) \quad (\text{eqn. of state})$$

$$\Rightarrow \delta P = \frac{\partial P}{\partial \rho}\delta\rho \equiv c_s^2\,\delta\rho \quad \left(c_s^2 \equiv \frac{\partial P}{\partial \rho}\right.$$

$= $ speed of sound in the fluid)

Hence, we have,

$$\ddot{\delta} + 2H\dot{\delta} - \frac{1}{a^2}\left(c_s^2\nabla^2\delta + 4\pi G\bar{\rho}a^2\delta\right) = 0$$

$$\Rightarrow \boxed{\ddot{\delta} + 2H\dot{\delta} - \left(\frac{c_s^2}{a^2}\nabla^2 + 4\pi G\bar{\rho}(t)\right)\delta = 0}$$

$\boxed{\delta P = c_s^2\,\delta\rho}$ $= c_s^2\bar{\rho}\delta$

and $\nabla^2\delta\Phi = 4\pi G\bar{\rho}a^2\delta$ is the 1st order Poisson eqn.

Hence, given $\bar{\rho}(t)$ and c_s, we can compute the time-evolution of the density contrast. This is the primary eqn. governing the growth of $\delta(t)$ for NR matter perturbations in the sub-Hubble regime.

Since, we are dealing with a linear eqn., it is convenient to use Fourier modes.

In Fourier space, $\delta(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec{k}\cdot\vec{x}}\,\tilde{\delta}(\vec{k},t)$

$$\ddot{\tilde{\delta}}(\vec{k},t) + 2H\dot{\tilde{\delta}}(\vec{k},t) + \left(\frac{c_s^2k^2}{a^2} - 4\pi G\bar{\rho}(t)\right)\tilde{\delta}(\vec{k},t) = 0$$

Each Fourier mode evolves independently at linear order.

define $$k_J(t) \equiv \frac{\sqrt{4\pi G\bar{\rho}(t)}}{c_s(t)}$$ | Physical Jeans wave-number

$$\Rightarrow 4\pi G\bar{\rho}(t) = c_s^2k_J^2$$

Hence, we obtain

$$\ddot{\tilde{\delta}} + 2H\dot{\tilde{\delta}} + c_s^2\left(\frac{k^2}{a^2} - k_J^2\right)\tilde{\delta} = 0$$

The Jeans scale in an expanding universe depends on time through $\bar{\rho}(t)$ and possibly $c_s(t)$.

Small scale perturbation ($\Rightarrow$ large k, wave-length $\lambda \sim \frac{1}{|\vec{k}|}$)

$$\frac{k}{a} \gg k_J$$

$$\Rightarrow c_s^2\left(\frac{k^2}{a^2} - k_J^2\right) > 0, \quad \equiv \omega^2 > 0$$

$\vec{k}$ is conjugate to $\vec{x}$, the co-moving length. Hence, $\vec{k}$ is the co-moving wave-number.

$$\vec{k}\cdot\vec{x} = \underbrace{\frac{\vec{k}}{a}}_{\text{physical wave number}} \cdot \underbrace{(a\vec{x})}_{\text{physical length}}$$

Hence,

$$\ddot{\delta} + 2H\dot{\delta} + \omega^2 \delta = 0$$

This is the eqn. of a damped harmonic oscillator, with a friction term due to the expansion, $2H\dot{\delta}$.

$\rightarrow$ Oscillatory solutions with decreasing amplitude.

Large scale perturbations: $\frac{k}{a} \ll k_J$, can ignore the pressure term, and obtain

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G \bar{\rho}(t)\, \delta = 0$$

Without the expansion damping term, the fluctuations would have grown exponentially!

Since this eqn. has no k-dependence, each Fourier mode time-evolves in the same way.

$$\delta(\vec{k}, t) = \delta_+(\vec{k})\, \underbrace{D_+(t)}_{\text{growing mode}} + \delta_-(\vec{k})\, \underbrace{D_-(t)}_{\text{decaying mode}}$$

(soln. to a 2nd order differential eqn.)

$\delta_\pm(\vec{k})$: Fourier modes of the initial density field

$D_\pm(t)$: $\vec{k}$-independent functions describing the linear time evolution

$D_+(t)$: linear growth function

Normalization: $D_+(t_0) \equiv 1$

if $c_s \to 0$ the pressure term $\to 0$, all modes will grow. $\rightarrow$ CDM.

More generally, $c_s^2 \to \langle v^2 \rangle$, and depending upon $\langle v^2 \rangle$ some modes may not grow with $\frac{k}{a} \gg k_J$ $\rightarrow$ suppression of power at small length scales $\rightarrow$ WDM or HDM

$\ln 10^6$
$= 6 \times \ln 10$
$= 6 \times 2.3$
$= 13.8$

$$\frac{T_{BBN}}{T_{MR}} = 10^6$$

Timeline:

$\overset{?}{\longleftarrow}$ T_{BBN} (1 MeV) T_{MR} (1 eV) T_{CMB} (0.26 eV) ...... T_0 (2.7 K)

Matter perturbations in a flat matter-dominated Universe: (Relevant for the growth of structure in our Universe after MR equality)

In MD : $a \propto t^{2/3} \Rightarrow H = \frac{2}{3t}$

$$\Rightarrow 4\pi G \bar{\rho}_m(t) = \frac{3}{2}H^2 \quad (\text{bkg. Friedmann eqn.})$$

$$= \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{4}\,2}{\cancel{9}_3 t^2} = \frac{2}{3t^2}$$

Hence, $\ddot{D} + \frac{4}{3t}\dot{D} - \frac{2}{3t^2}D = 0$

Try the ansatz $D \sim t^\alpha$

then $\alpha(\alpha-1)\,t^{\alpha-2} + \frac{4}{3t}\alpha t^{\alpha-1} - \frac{2}{3t^2}t^\alpha = 0$

$\Rightarrow \alpha(\alpha-1) + \frac{4\alpha}{3} - \frac{2}{3} = 0$

$\Rightarrow 3\alpha^2 + \alpha - 2 = 0$

$\Rightarrow \alpha = \frac{-1 \pm \sqrt{1+24}}{6} = \frac{-1\pm 5}{6} = -1, 2/3$

Hence, $D \sim t^{-1}$ (decaying mode)

or $D \sim t^{2/3}$ (growing mode)

Hence, $D_+(t) \propto t^{2/3}$

$\Rightarrow D_+ \propto a$

and $D_- \propto a^{-3/2}$

(*) (*) Hence the density contrast for modes with $\frac{k}{a} \ll k_J$ grows as $\boxed{\delta \propto a}$, proportional to the scale factor.

But $\delta\rho$, the density fluctuation, is still decreasing, $\delta\rho = \bar{\rho}\,\delta \propto \frac{1}{a^3} \times a \propto a^{-2}$, which is at a rate slower than the bkg. density $\bar{\rho} \propto a^{-3}$.

Now, $\nabla^2 \delta\Phi = 4\pi G a^2 \bar{\rho}\,\delta$.

as $\bar{\rho}\,\delta \propto a^{-2}$, $\nabla^2\delta\Phi$ is independent of time.

$\Rightarrow \delta\Phi$ is a constant in time in the matter era.

$c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)$ (For a relativistic gas $p \simeq \frac{1}{3}\rho$ $\Rightarrow \left(\frac{\partial p}{\partial \rho}\right) = \frac{1}{3} \Rightarrow c_s = \frac{1}{\sqrt{3}}$)

The $(k/a \ll k_J)$ discussion above applies to baryons after they decouple from photons. Before recombination, the baryons are strongly coupled to photons, forming a (relativistic) photon-baryon fluid, with a sound speed $c_s \sim \frac{1}{\sqrt{3}}$. Then, if I assume the above eqns to still hold up to relativistic corrections, $k_J \sim \sqrt{12\pi G \rho} = \sqrt{3H^2 \times \frac{3}{2}} \sim 2H$.

For growing modes, $\frac{k}{a} < k_J$

$$\Rightarrow \frac{k}{a} < H \Rightarrow \lambda_{physical} > H^{-1}$$

all sub-Hubble modes oscillate.

Only super-Hubble modes may grow, but all sub-Hubble modes oscillate. The above Newtonian analysis is anyway invalid for super-Hubble modes.

For sub-Hubble modes, $\frac{a}{k} < H^{-1}$ $\Rightarrow \frac{a}{k} < k_J^{-1}$ $\Rightarrow \boxed{\frac{k}{a} > k_J}$ $\boxed{\text{oscillatory}}$

$\Rightarrow$ Sub-Hubble modes grow only after recombination for baryons, as they decouple from photons, and c_s reduces.

For the no pressure case, $c_s^2 \to \langle v^2 \rangle$

(*) If on the other hand we had a NR fluid (CDM) or baryons not coupled to photons, for them $\boxed{k_J \gg H}$, as $c_s \ll 1$.
For sub-Hubble modes, $\lambda_{physical} < H^{-1}$

$$\Rightarrow \boxed{\frac{k}{a} > H}$$

$\Rightarrow$ ~~$\frac{k}{a} > k_J$~~ there will be modes for which $\boxed{\frac{k}{a} \ll k_J} \to$ growth.

Hence baryonic perturbations can grow only from T_{CMB}, while CDM perturbation grow linearly from T_{MR}. They also grow logarithmically during RD.

How do the perturbations for such a decoupled NR fluid grow during RD?

$$\ddot{\delta}_c + 2H\dot{\delta}_c - 4\pi G \sum_a \bar{\rho}_a \delta_a = 0$$

$$\sum_a \bar{\rho}_a \delta_a = \bar{\rho}_c \delta_c + \bar{\rho}_\gamma \delta_\gamma$$

↳ includes the baryon-radiation fluid.

For the radiation fluid, c_s is large, the pressure wave travels fast and hence δ_γ oscillates on scales smaller than the horizon.

$\Rightarrow$ time averaged density contrast of radiation vanishes. DM (δ_c) is the only clustered component. Hence,

$$\ddot{\delta}_c + 2H\dot{\delta}_c - 4\pi G \bar{\rho}_c \delta_c = 0$$

Logarithmic growth during RD : 13.8 a factor

Linear " " MD upto 0.26 eV = factor 4.

$\Rightarrow$ together they give a factor > 55

Even if I start from $\delta_c \sim 5\times10^{-5}$ $\downarrow$ $\delta_c \sim 275\times10^{-5}$ @ $z \simeq 1100$ $\downarrow$ $\delta_c \simeq 273\times10^{-3} \simeq 0.273$

But if $\frac{\Delta T}{T} \sim 5\times10^{-5}$ then $\frac{\Delta\rho}{\rho} = 4\frac{\Delta T}{T}$ which will take δ_c to $O(1)$ values by $z=10$

Of course, the homogeneous radiation density still plays an imp. role in determining the Hubble expansion rate, $H = \frac{1}{2t}$ for $a(t) \propto t^{1/2}$.

(as $\frac{1}{t} \sim H$)

$$\ddot{\delta}_c \approx H^2 \delta_c \gg 4\pi G \bar{\rho}_c \delta_c$$ as H^2 is determined by $\bar{\rho}_r$ in RD, and $\bar{\rho}_r \gg \bar{\rho}_c$ in RD. Hence, in $\ddot{\delta}_c + 2H\dot{\delta}_c - 4\pi G\bar{\rho}_c\delta_c = 0$, we can drop $4\pi G\bar{\rho}_c\delta_c$ w.r.t. $\ddot{\delta}_c$, hence,

$$\ddot{\delta}_c + 2H\dot{\delta}_c = 0$$

$$\text{or,} \quad \ddot{D} + \frac{1}{t}\dot{D} = 0$$

Two solutions: $D_-(t) \propto$ constant

or $D_+(t) \propto \ln t \propto \ln a$

$\Rightarrow$ The perturbations δ_c can only grow logarithmically in the RD era, which spans at least from $T_{BBN} \simeq 10^6$ eV to $T_{MR} \sim 1$ eV !

(*) If something like dark energy exists and dominates the energy density in some late epoch, then, since it does not cluster, it does not contribute to $\nabla^2 \delta\Phi$. Hence,

$\delta_{DE} = 0$ as it does not cluster

$$\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G\bar{\rho}_m\delta_m = 0$$

$\left(H^2 = \frac{8\pi G}{3}\bar{\rho}_{DE} \text{ '' constant} \right)$

$H \approx$ constant during DE domination, furthermore, $4\pi G\bar{\rho}_m\delta_m \ll \ddot{\delta}_m$, we have therefore

$$\ddot{\delta}_m + 2H\dot{\delta}_m = 0$$

~~$D(t) =$ constant, or $D(t) \propto \frac{1}{a^2}$~~

~~$\frac{d}{dt}a^{-2} = -2a^{-3}\dot{a}$~~

~~$\frac{d^2}{dt^2}a^{-2} = -2\frac{d}{dt}(a^{-3}\dot{a})$~~

~~$= -2\ddot{a}a^{-3}$~~

$\Rightarrow \frac{d}{dt}\left(\dot{\delta}_m + 2H\delta_m\right) = 0$ as $H =$ constant

$\Rightarrow \dot{\delta}_m + 2H\delta_m =$ constant

$\Rightarrow a\dot{\delta}_m + \frac{2\dot{a}}{a}\delta_m =$ '' $\Rightarrow$ ~~$\frac{d}{dt}(\delta_m a) =$ constant~~

Hence, the eqn. for the growth function is

$$\ddot{D} + 2H\dot{D} = 0 \quad , \text{ with } H \sim \text{const.}$$

$\Rightarrow \frac{\dot{a}}{a} = \text{const}$

$\Rightarrow \frac{da}{a} = \text{const} \times dt$

$\Rightarrow \ln a = \text{const} \times t$

$\Rightarrow a = e^{Ht}$

$\frac{\dot{a}}{a} = \frac{H e^{Ht}}{e^{Ht}} = H.$

$\dot{a} = Ha$

$\Rightarrow \ddot{a} = H\dot{a} = aH^2$

Ansatz: $D = a^{\alpha}$

$\Rightarrow \dot{D} = \alpha a^{\alpha-1} \dot{a}$

$\Rightarrow \ddot{D} = \alpha \ddot{a} a^{\alpha-1} + \alpha(\alpha-1) \dot{a}^2 a^{\alpha-2}$

Hence,

$$\alpha \ddot{a} a^{\alpha-1} + \alpha(\alpha-1) a^{\alpha-2} \dot{a}^2 + 2H\alpha a^{\alpha-1} \dot{a} = 0$$

$\Rightarrow$ either $\alpha = 0$, or, $D = \text{constant}$

or

$$\ddot{a} + (\alpha-1) a^{-1} \dot{a}^2 + 2H\dot{a} = 0$$

$$\Rightarrow aH^2 + (\alpha-1) a^{-1} a^2 H^2 + 2H^2 a = 0$$

$$\Rightarrow aH^2 (3 + \alpha - 1) = 0$$

$$\Rightarrow \alpha = 1 - 3 = -2$$

Hence $D(t) \propto \begin{cases} a^{-2} & \rightarrow D_-(t) \\ \text{const.} & \rightarrow D_+(t) \end{cases}$

Hence, the growth of structure halts in the DE era.

Behaviour of different modes: K-dependence of growth

Hubble radius $\sim H^{-1}$ (sometimes also loosely referred to as the horizon, which is of the same order).

As the Universe expands, H changes:

$$H^2 = \frac{8\pi G}{3} \bar{\rho} \propto a^{-3} \text{ in M.D.}$$
$$\propto a^{-4} \text{ in R.D.}$$

$$\Rightarrow H \propto a^{-3/2} \text{ in M.D.}$$
$$\propto a^{-2} \text{ in R.D.}$$

$$\text{or, } H^{-1} \propto a^{3/2} \text{ in M.D.}$$
$$\propto a^{2} \text{ in R.D.}$$

Hence the Hubble radius grows with time.

k : co-moving wave-number

$\lambda = \frac{1}{k}$: " wave-length

$\lambda_{physical} = \frac{a}{k}$: physical wave-length of a Fourier mode.

$\lambda_{physical}$ also grows with time (k is fixed)

$\Rightarrow \frac{k}{a}$: decreases w/ time as the Universe expands.

If $\lambda_{physical} \sim H^{-1}$

$\Rightarrow \frac{a}{k} \sim H^{-1} \Rightarrow \boxed{k \sim aH}$

If $\lambda_{physical} > H^{-1}$ (Super-Hubble, super Horizon)

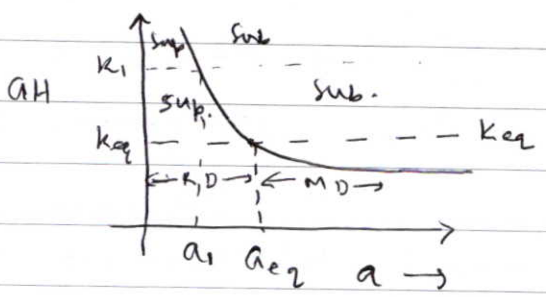
$\Rightarrow \frac{a}{k} > H^{-1} \Rightarrow \boxed{k < aH}$

If $\lambda_{physical} < H^{-1}$ (Sub-Hubble, sub-Horizon)

$\Rightarrow \frac{a}{k} < H^{-1} \Rightarrow \boxed{k > aH}$

Now, $aH \propto a^{-1/2}$ in M.D.

$\propto a^{-1}$ in R.D.



$K_{eq} = a_{eq} H(a_{eq})$

(The mode k_{eq} enters the horizon at a_{eq})

For the value of the scale factor a_1, s.t. $k_1 = a_1 H_1$, the modes k_1 enters the Horizon. For $a > a_1$, the mode is sub-Hubble, and for $a < a_1$, the mode is super-Hubble. So, at an early enough time, all modes should be super-Hubble, and at a late enough time, all modes should eventually enter the Horizon.

Dynamics of perturbations at super-Hubble scales require general relativistic perturbation theory, as space-time curvature effects become relevant.
In the comoving gauge, the superhorizon evolution of δ is:

gauge-dependent

$$\delta(\vec{k},t) \propto \frac{1}{(aH)^2} \propto \begin{cases} a^2 & \text{in R.D.} \\ a & \text{in M.D.} \end{cases}$$

Hence in M.D., the density contrast grows as $a(t)$ both inside and outside the horizon.
For modes entering the horizon in the MD, the growth is thus independent of scale.
In contrast, in RD, the perturbations behave differently in- and out-side the horizon.
Sub-horizon modes $\rightarrow$ experience logarithmic growth

Sup- " " $\rightarrow$ grow as a^2.

The moment of horizon entry depends on the wave-number of the mode, $k = aH$, which leads to a k-dependent growth of the fluctuations.

Transfer Function: $T(k) \equiv \dfrac{D_+(t_i)}{D_+(t)} \dfrac{\delta(\vec{k},t)}{\delta(\vec{k},t_i)}$

D_+ : scale-independent growth function

t_i : initial time when all modes of interest are outside the horizon.

k-dependence of the growing mode
$\delta(\vec{k},t) = \delta_+(\vec{k})\, D_+(t)$
$\Rightarrow \dfrac{D_+(t)}{\delta(\vec{k},t)} = \delta_+(\vec{k})$ at t
$\dfrac{D_+(t_i)}{\delta(\vec{k},t_i)} = \delta_+(\vec{k})$ at t_i

$k_{eq} \equiv (aH)_{eq}$: wave-number of the mode that entered the horizon at MR equality.

For $k > k_{eq}$, the modes enter the horizon at RD
For $k < k_{eq}$, " " " " MD

$\Rightarrow$ The growth is scale-independent for $k < k_{eq}$.

$$k_{eq} = a_{eq} H(a_{eq}) \simeq 0.015\, h \ \mathrm{Mpc}^{-1}$$
$$\simeq 0.0105 \ \mathrm{Mpc}^{-1} \quad \text{for } h = 0.7$$

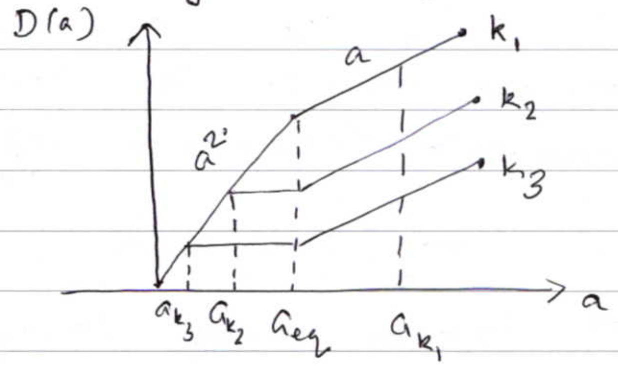
$$\delta(\vec{k}, t) = \delta_+(\vec{k})\, D_+(t)$$
$$\delta(\vec{k}, t_i) = \delta_+(\vec{k})_i\, D_+(t_i)$$

$$T(k) = \frac{\delta_+(\vec{k})}{\delta_+(\vec{k})_i} = \frac{D_+(t_i)}{D_+(t)} \frac{\delta(\vec{k},t)}{\delta(\vec{k},t_i)}$$

[given initial $\delta_+(\vec{k}_i)$, the final one $\delta_+(\vec{k})$ can be obtained using $T(k)$:
$\delta_+(\vec{k}) = T(k)\, \delta_+(\vec{k})_i$

Consider long-wave length mode : $k < k_{eq}$

(the logarithmic growth is not shown here)



(*) k_1 enters horizon at MD at a_{k_1}. Hence upto a_{eq} it grows as a^2 and after that as a.

at a_{eq}, $$\frac{\delta(\vec{k}, t_{eq})}{a_{eq}^2} = \frac{\delta(\vec{k}, t_i)}{a_i^2}$$

and, $$\frac{\delta(\vec{k}, t)}{a} = \frac{\delta(\vec{k}, t_{eq})}{a_{eq}} = \frac{\delta(\vec{k}, t_i)}{a_i^2} a_{eq}^2 \frac{1}{a_{eq}}$$

$$\Rightarrow \delta(\vec{k}, t) = \delta(\vec{k}, t_i) \left(\frac{a_{eq}}{a_i}\right)^2 \left(\frac{a}{a_{eq}}\right)$$
$$\equiv \frac{D_+(t)}{D_+(t_i)} \cancel{D_+(t_{eq})}\ \delta(\vec{k}, t_i)$$

$$T(k < k_{eq}) = 1$$

<u>Short-wavelength modes:</u> $k > k_{eq}$ (e.g. k_2, k_3)

Before horizon entry at a_{k_2} or a_{k_3}, $\delta \sim a^2$.

After " " until a_{eq}, $\delta \sim \ln a \sim$ const. approximately

After a_{eq}, $\delta \sim a$.

$$\delta(\vec{k}, t) = \left(\frac{a_k}{a_i}\right)^2 \left(\frac{a}{a_{eq}}\right) \delta(\vec{k}, t_i)$$

$$= \underbrace{\left(\frac{a_k}{a_{eq}}\right)^2}_{\substack{\text{additional} \\ \text{suppression} \\ \text{for } k > k_{eq}}} \times \underbrace{\left[\left(\frac{a_{eq}}{a_i}\right)^2 \left(\frac{a}{a_{eq}}\right)\right]}_{\text{Same as before}} \delta(\vec{k}, t_i)$$

$H \propto a^{-2}$ in RD, so k enters the horizon at a_k, s.t. $k = (aH)_k$

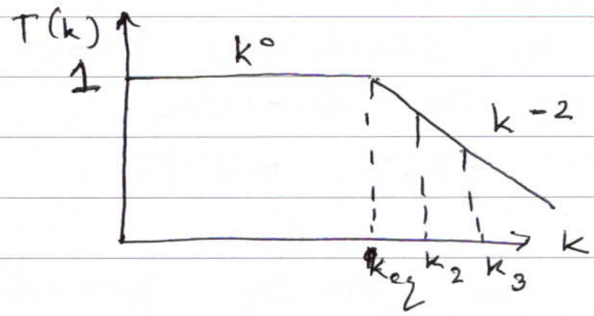
$$\propto a_k a_k^{-2} \propto \frac{1}{a_k}$$

Hence, $\left(\frac{a_k}{a_{eq}}\right)^2 = \left(\frac{k_{eq}}{k}\right)^2$

Hence, $T(k > k_{eq}) = \left(\frac{k_{eq}}{k}\right)^2 \left[\times \ln\left(\frac{k}{k_{eq}}\right)\right]$

$T(k < k_{eq}) = 1$

$$P(k) \sim |\delta_k|^2 \sim T^2(k) \frac{D_+^2(t)}{D_+^2(t_i)} P(k, t_i)$$



From magnitude to correlations

Statistical properties:

$\boxed{\text{(*) The large-scale structure in the Universe isn't distributed randomly, but has interesting correlations between spatially separated points.}}$

(*) Not just the degree of the density fluctuations, but correlations between them are important.

(*) $\langle \delta \rangle = 0$ $\quad$ ($\langle\ \rangle$: (ensemble average $= \bar{\delta}$ = spatial average in infinite vol. limit))

(*) 2-pt. correlation function

$$\xi(\vec{x}_1, \vec{x}_2) \equiv \langle \delta(\vec{x}_1)\, \delta(\vec{x}_2) \rangle$$

Statistical homogeneity: although the Universe is inhomogeneous, it is statistically homogeneous, i.e., the expectation value $\langle f(\vec{x}) \rangle$ must be the same at all $\vec{x}$.

$\Rightarrow$ $\xi(\vec{x}_1, \vec{x}_2)$ can depend only on the separation $\vec{r} = \vec{x}_2 - \vec{x}_1$. Hence, re-define ξ:

$$\xi(\vec{r}) \equiv \langle \delta(\vec{x})\, \delta(\vec{x} + \vec{r}) \rangle$$

Statistical isotropy: For quantities that involve a direction, the statistical properties are independent of the direction.

$$\Rightarrow \quad \xi(\vec{r}) = \xi(r), \quad r = |\vec{r}|$$

(*) Variance of the density perturbation:

$$\langle \delta^2 \rangle \equiv \langle \delta(\vec{x})\, \delta(\vec{x}) \rangle \equiv \xi(0).$$

Fourier space:

$$\delta(\vec{x}) = \frac{1}{(2\pi)^3} \int \delta(\vec{k})\, e^{i\vec{k}\cdot\vec{x}}\, d^3k$$

$$\delta(\vec{k}) = \int \delta(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, d^3x$$

$$\boxed{\begin{array}{l} \delta(\vec{x})^* = \delta(\vec{x}) \\ \delta^*(\vec{k}) = \delta(-\vec{k}) \end{array}}$$

$$\langle \delta(\vec{k}) \rangle = 0$$

What is $\langle \delta^*(\vec{k})\, \delta(\vec{k}') \rangle$?

$$= \int d^3x\, e^{i\vec{k}\cdot\vec{x}} \int d^3x'\, e^{-i\vec{k}'\cdot\vec{x}'} \langle \delta(\vec{x})\, \delta(\vec{x}') \rangle$$

$$= \int d^3x\, e^{i\vec{k}\cdot\vec{x}} \int d^3r\, e^{-i\vec{k}'\cdot(\vec{x}+\vec{r})} \langle \delta(\vec{x})\, \delta(\vec{x}+\vec{r}) \rangle \qquad \boxed{\vec{x}' = \vec{x} + \vec{r}}$$

$$= \int d^3r\, \xi(\vec{r})\, e^{-i\vec{k}'\cdot\vec{r}} \int d^3x\, e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}$$

$$= (2\pi)^3\, \delta^{(3)}(\vec{k}-\vec{k}')\, P(\vec{k}), \text{ where,}$$

$$P(\vec{k}) = \int d^3r\, \xi(\vec{r})\, e^{-i\vec{k}\cdot\vec{r}} \qquad (\text{power spectrum})$$

(*) statistical homogeneity $\Rightarrow$ $\delta(\vec{k})$ are uncorrelated.

(Assumed $\xi(\vec{r}) \to 0$ for $|\vec{r}| \to \infty$)

$$\xi(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3k\, e^{i\vec{k}\cdot\vec{r}}\, P(\vec{k})$$

As $\xi(\vec{r}) = \xi(r)$, $\Rightarrow P(\vec{k}) = P(k)$

$$P(k) = \int_0^\infty \xi(r)\, \frac{\sin kr}{kr}\, 4\pi r^2\, dr.$$

Check: $P(k) = \int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\, \xi(r)$

$$= \int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\theta) \int_0^\infty dr\, r^2\, e^{-ikr\cos\theta}\, \xi(r)$$

$$\xi(r) = \frac{1}{(2\pi)^3} \int_0^\infty P(k)\, \frac{\sin kr}{kr}\, 4\pi k^2\, dk.$$

The density variance

$$\langle \delta(\vec{x})\,\delta(\vec{x}) \rangle = \langle \delta^2 \rangle = \xi(0)$$

$$= \frac{1}{(2\pi)^3} \int_0^\infty P(k)\, 4\pi k^2\, dk$$

$$= \frac{1}{2\pi^2} \int_0^\infty k^3\, P(k)\, \frac{dk}{k}$$

$$= \frac{1}{2\pi^2} \int_{-\infty}^{\infty} k^3\, P(k)\, d\ln k$$

$$\equiv \int_{-\infty}^{\infty} \mathcal{P}(k)\, d\ln k$$

where, we have defined

$$\boxed{\mathcal{P}(k) \equiv \frac{k^3}{2\pi^2} P(k)}$$

(another notation for $\mathcal{P}(k)$ is $\Delta^2(k)$)

Power per logarithmic interval of scales to the density variance.

$\mathcal{P}(k)$ is dimensionless.

$$[P(k)] = \frac{1}{[k^3]} = [L]^3 \rightarrow \text{Mpc}^3.$$

We had

$$P(k,t) = T^2(k)\, \frac{D_+^2(t)}{D_+^2(t_i)}\, P(k,t_i)$$

If the primordial power spectrum is of the form $P(k,t_i) = A k^n$, with A, n constants (at least in some interval)

n: spectral index.

If further $n \sim 1$, then by the Poisson eqn., $\nabla^2 \delta\Phi = 4\pi G a^2 \bar{\rho}\, \delta$

we have $\delta\Phi_k = \underbrace{\frac{4\pi G a^2}{k^2} \bar{\rho}\, \delta}$,

then the power spectrum of $\delta\Phi_k$

$$P_{\delta\Phi}(k, t_i) \propto k^{-4} P_\delta(k, t_i) \propto k^{n-4}$$

Hence, the variance of the gravitational potential

$$\sigma_\Phi^2 \equiv \xi_\Phi(0) = \int \frac{dk}{k} \frac{k^3}{2\pi^2} P_\Phi(k, t_i)$$

$$\equiv \int \frac{dk}{k} \Delta^2_\Phi(k)$$

where the dim.less power spectrum

$$\Delta^2_\Phi(k) \equiv \frac{k^3}{2\pi^2} P_\Phi(k, t_i) \propto k^{n-1}$$

Hence, for $n = 1$, the dim.less power spectrum of the gravitational potential becomes k-independent, so that the variance receives equal contributions from every decade in k.
→ scale invariance.

CMB observations indicate $n \sim 1$ $(n = 0.9667 \pm 0.0040)$

We had

$$T(k) \sim \begin{cases} 1 & \text{for } k < k_{eq} \\ \left(\frac{k_{eq}}{k}\right)^2 \left[\times \ln \frac{k}{k_{eq}}\right], & \text{for } k > k_{eq}. \end{cases}$$

And if we "postulate"

$$P(k, t_i) = A k^n$$

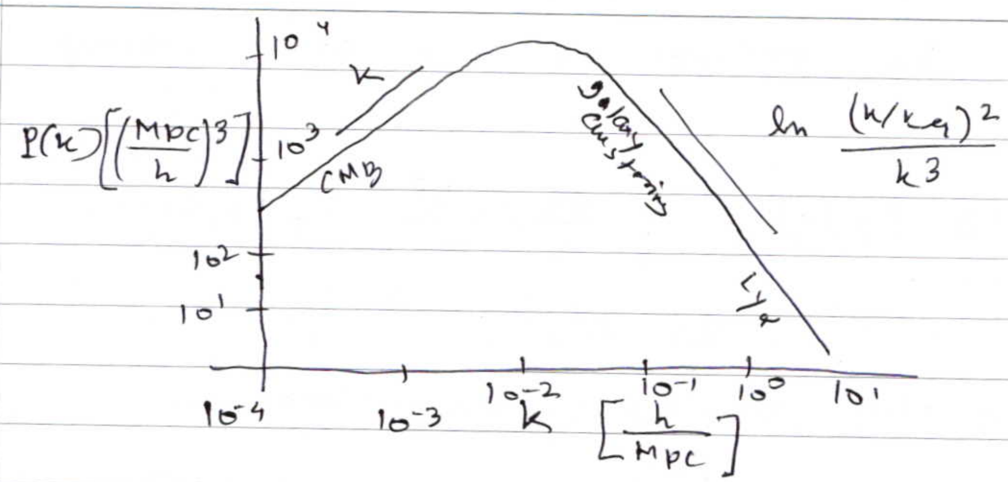
then,

$$P(k,t) \propto \begin{cases} k^n & \text{for } k < k_{eq} \\ k^{n-4} \left(\ln \frac{k}{k_{eq}}\right)^2 & \text{for } k > k_{eq} \end{cases}$$

For $n = 1$, we therefore have

$$P(k,t) \propto \begin{cases} k & \text{for } k < k_{eq} \text{ (large scales)} \\ k^{-3} \left(\ln \frac{k}{k_{eq}}\right)^2 & \text{for } k > k_{eq} \text{ (small scales)} \end{cases}$$

Recall $k_{eq} \simeq 0.015\, h\ \mathrm{Mpc}^{-1}$



Aside:

=0=

Baryon in-fall in DM wells:

After decoupling:

$$\ddot{\delta}_c + \frac{4}{3t}\dot{\delta}_c = 4\pi G\left(\bar{\rho}_b \delta_b + \bar{\rho}_c \delta_c\right)$$

$$\ddot{\delta}_b + \frac{4}{3t}\dot{\delta}_b = 4\pi G\left(\bar{\rho}_b \delta_b + \bar{\rho}_c \delta_c\right)$$

Subtracting, $\ddot{\Delta} + \frac{4}{3t}\dot{\Delta} = 0$

where $\Delta = \delta_c - \delta_b$. Try a power law ansatz:

$\Delta \sim t^n$

Sols: $n = 0$, $n = -1/3$.

Growing mode: $\Delta \sim$ constant, but $\delta_m \propto a$.

$$1 - \frac{\delta_b}{\delta_c} = \frac{\delta_c - \delta_b}{\delta_c} = \frac{\Delta}{\delta_c} \to \frac{1}{a} \to 0$$

for large a.

$\Rightarrow \frac{\delta_b}{\delta_c} \to 1$ at late times.