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ADDENDUM TO "A NOTE ON THE DISTRIBUTION OF VALUES OF AN ARITHMETICAL FUNCTION"

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Some informations about the distribution function

$$V(\lambda) \stackrel{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N} \sum_{n \le N,\, \delta_k(n) < n\lambda} 1$$

where $k = p_1 p_2 \ldots p_r$ is a product of distinct prime numbers and

$$\delta_k(n) = \max \{d : d \mid n, (d, k) = 1\},$$

are derived here from the theorem (stated here as Theorem 3) proved in the paper "A note on the distribution of values of an arithmetical function", *Indian J. pure appl. Math.*, **26** (1995), 931-35, written by the authors. Since these observations complete the picture in some sense and are finer than what could have been obtained from the classical theorems on the subject, we felt the need for making them explicit. We also take this opportunity to point out further problems in this direction as well as to compare our results with those obtained classically in the case of Euler's phi function.

§1. First we state the following celebrated theorems due to Erdös and Wintner as we shall have to mention them in the sequel.

Theorem 1 (Erdös-Wintner[2]) — For any additive function $f(n)$, convergence of the series

$$\sum_{|f(p)| \ge 1} \frac{1}{p}, \quad \sum_{|f(p)| < 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| < 1} \frac{f^2(p)}{p}$$

is necessary and sufficient for weak convergence of the sequence of distribution functions

$$F_N(x) = \frac{1}{N} \sum_{n \le N, f(n) \le x} 1$$

as $N \to \infty$.

Theorem 2 (Erdös-Wintner[2]) — A distribution function that is the asymptotic distribution function for an additive function $f(n)$ is continuous if and only if the series $\Sigma_{f(p)\neq 0} 1/p$ diverges.

We consider the distribution function $V(\lambda)$ defined by

$$V(\lambda) \overset{\text{def}}{=} \lim_{N\to\infty} \frac{1}{N} \sum_{n\leq N,\, \delta_k(n)<n\lambda} 1$$

where $k = p_1p_2 \dots p_r$ is a product of distinct prime numbers and

$$\delta_k(n) = \max\{d : d \mid n,\ (d, k) = 1\}.$$

Regarding $V(\lambda)$, the following theorem was proved by the authors[1].

Theorem 3 —

$$V(\lambda) = \frac{\phi(k)}{k} \sum_{\substack{n=1 \\ \delta_k(n)<n\lambda}}^{\infty} \frac{\chi_k(n)}{n} = \frac{\phi(k)}{k} \sum_{\substack{n=1 \\ \delta_k(n)<n\lambda}}^{x} \frac{\chi_k(n)}{n} + O\left(\frac{\log^{r-1}(x)}{x}\right)$$

where

$$\chi_k(n) = \begin{cases} 1, & \text{if every prime divisor of } n \text{ divides } k \\ 0, & \text{otherwise.} \end{cases}$$

Since $\delta_k(n)/n$ is multiplicative, $f(n) = \log(\delta_k(n)/n)$ is an additive function. Also, from the definition of $\delta_k(n)$, it is clear that $f(n) = 0$ whenever k and n are relatively prime. Therefore, the conditions of Theorem 1 are clearly satisfied and hence the existence of $V(u)$ follows.

Further, since $f(p) = \log(\delta_k(p)/p) = 0$ whenever p does not divide k,

$$\sum_{f(p)\neq 0} 1/p = \sum_{p\mid k} 1/p < \infty$$

and one concludes from Theorem 2 that the asymptotic distribution function for $\log(\delta_k(n)/n)$ (and hence that for $\delta_k(n)/n$) is not continuous.

We point out that from Theorem 3, one can derive more precise informations about $V(\lambda)$.

We observe that $\chi_k(n) = 1$ whenever n is of the form $n = p_1^{a_1} \dots p_r^{a_r}$, $a_i \geq 0$. But for these n, $\delta_k(n) = 1$. Therefore we can rewrite Theorem 3 as follows.

$$V(\lambda) = \frac{\phi(k)}{k} \sum_{n>\frac{1}{\lambda}} \frac{\chi_k(n)}{n}$$

$$= \frac{\phi(k)}{k}\left(\sum_{n=1}^{\infty} \frac{\chi_k(n)}{n} - \sum_{n\leq\frac{1}{\lambda}} \frac{\chi_k(n)}{n} \right)$$

$$= 1 - \frac{\phi(k)}{k} \sum_{n \leq \frac{1}{\lambda}} \frac{\chi_k(n)}{n} \left(\text{since} \sum_{n=1}^{\infty} \frac{\chi_k(n)}{n} = \prod_{p \mid k} \left(1 - \frac{1}{p}\right)^{-1} = \frac{k}{\phi(k)} \right).$$

Thus, $V(\lambda)$ is a monotone step function and the points of discontinuity are precisely the points

$$\left\{ \frac{1}{n} : 1 < n = \prod_{i=1}^{r} p_i^{a_i}, a_i \geq 0 \right\}.$$

§2. Our investigations were, to some extent, motivated by some results obtained in the case of Euler's phi function, which by now, are classical. But in that case, writing

$$V'(\lambda) \overset{\text{def}}{=} \lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N, \, \phi(n) < n\lambda} 1 \quad (0 < \lambda < 1)$$

the distribution function $V'(\lambda)$ is continuous.

In fact, it is further known (Venkov[5]) that

(1) $V'(\lambda)$ is an increasing function on [0, 1], i.e., if $1 > \lambda_1 > \lambda_2 > 0$, then $V'(\lambda_1) > V'(\lambda_2)$.

(2) $V'(\lambda)$ has a left derivative equal to infinity on an everywhere dense set of points in [0, 1].

A further investigation in the case of $\delta_k(n)$ is suggested by the papers of Tyan[4] and Faǐnleǐb[3] where they deal with the question of the rate at which the sequence of functions

$$\frac{1}{N} \sum_{\phi(n) < n\lambda, \, n \leq N} 1 = \Phi_N(\lambda)$$

approaches its limit as $N \to \infty$. The theorem of Faǐnleǐb is rather deep. We point out that the second part of our Theorem 3 answers a different question of the similar nature; the difficulties here went unnoticed as we were able to make use of some deep results on the estimation of the number of integer lattice points in tetrahedrons.

References

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