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Liouville numbers and Schanuel's Conjecture

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Abstract. In this paper, using an argument of P. Erdős, K. Alniaçik, and É. Saias, we extend earlier results on Liouville numbers, due to P. Erdős, G.J. Rieger, W. Schwarz, K. Alniaçik, É. Saias, E.B. Burger. We also produce new results of algebraic independence related with Liouville numbers and Schanuel's Conjecture, in the framework of G_δ-subsets.

1. Introduction. For any integer q and any real number $x \in \mathbf{R}$, we denote by

$$\|qx\| = \min_{m\in\mathbf{Z}} |qx - m|$$

the distance of qx to the nearest integer. Following Maillet [8,9] an irrational real number ξ is said to be a *Liouville number* if, for each integer $n \geq 1$, there exists an integer $q_n \geq 2$ such that the sequence $\big(u_n(\xi)\big)_{n\geq 1}$ of real numbers defined by

$$u_n(\xi) = -\frac{\log \|q_n\xi\|}{\log q_n}$$

satisfies $\lim_{n\to\infty} u_n(\xi) = \infty$. If p_n is the integer such that $\|q_n\xi\| = |q_n\xi - p_n|$, then the definition of $u_n(\xi)$ can be written

$$|q_n\xi - p_n| = \frac{1}{q_n^{u_n(\xi)}}\cdot$$

An equivalent definition is to say that a Liouville number is a real number ξ such that, for each integer $n \geq 1$, there exists a rational number p_n/q_n with $q_n \geq 2$ such that

$$0 < \left|\xi - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^n}\cdot$$

We denote by $\mathbb{L}$ the set of Liouville numbers. This set $\mathbb{L}$ is an uncountable, dense subset of $\mathbf{R}$ having Lebesgue measure 0 and

$$\mathbb{L} = \bigcap_{n\geq 1} U_n \quad \text{with} \quad U_n = \bigcup_{q\geq 2} \bigcup_{p\in \mathbf{Z}} \left(\frac{p}{q} - \frac{1}{q^n},\ \frac{p}{q} + \frac{1}{q^n} \right) \setminus \left\{ \frac{p}{q} \right\}.$$

Each U_n is dense since each $p/q \in \mathbf{Q}$ belongs to the closure of U_n. Throughout this article, a G_δ-subset of a topological space X is defined to be the countable intersection of dense open subsets of X. Baire's theorem states that in a complete or locally compact space X, any G_δ-subset is dense. Since $\mathbf{R}$ is complete, we see that $\mathbb{L}$ is a G_δ-subset of $\mathbf{R}$. The G_δ-subset is also defined as a set having a complement which is *meager*. In our case, this complement $\mathbb{L}^c$ is the set of non–Liouville numbers

$$\mathbb{L}^c = \Bigg\{ x \in \mathbf{R} \ \Big| \ \text{there exists } \kappa > 0 \text{ such that} \\ \left| x - \frac{p}{q} \right| \geq \frac{1}{q^\kappa} \text{ for all } \frac{p}{q} \in \mathbf{Q} \text{ with } q \geq 2 \Bigg\},$$

which has full Lebesgue measure.

In 1844, Liouville [7] proved that any element of $\mathbb{L}$ is a transcendental number. A survey on algebraic independence results related with Liouville numbers is given in [13].

In 1962, Erdős [6] proved that *every real number t can be written as $t = \xi + \eta$ with ξ and η Liouville numbers*. He gave two proofs of this result. The first one is elementary and constructive: he splits the binary expansion of t into two parts, giving rise to binary expansions of two real numbers ξ and η, the sum of which is t. The splitting is done in such a way that both binary expansions of ξ and η have long sequences of 0's. The second proof is not constructive as it relies on Baire's Theorem. In the same paper, Erdős gives also in the same way two proofs, a constructive one and another depending on Baire's Theorem, that *every non-zero real number t can be written as $t = \xi\eta$, where ξ and η are in $\mathbb{L}$*. From each of these proofs, it follows that there exist uncountably many such representations $t = \xi + \eta$ (resp. $t = \xi\eta$) for a given t. Many authors extended this result in various ways: Rieger in 1975 [10], Schwarz in 1977 [11], Alniaçik in 1990 [1], Alniaçik and Saias in 1994 [2], Burger in 1996 [4] and 2001 [5]. In [4], Burger extended Erdős's result to a very large class of functions, including $f(x, y) = x + y$ and $g(x, y) = xy$.

Recall [2] that a real function $f : \mathcal{I} \to \mathbf{R}$ is *nowhere locally constant* if, for every nonempty open interval $\mathcal{J}$ contained in $\mathcal{I}$, the restriction to $\mathcal{J}$ of f is not constant. We define in a similar way a function which is *nowhere locally zero*: for every nonempty open interval $\mathcal{J}$ contained in $\mathcal{I}$, the restriction to $\mathcal{J}$ of f is not the zero function.

The main result of [2], which extends the earlier results of [10] and [11], deals with G_δ-subsets, and reads as follows (see [3, Exercise 1.6]):

Proposition 1. (Alniaçik–Saias) *Let $\mathcal{I}$ be an interval of $\mathbf{R}$ with nonempty interior, G a G_δ-subset of $\mathbf{R}$, and $(f_n)_{n\geq 0}$ a sequence of real maps on $\mathcal{I}$ which are continuous and nowhere locally constant. Then*

$$\bigcap_{n\geq 0} f_n^{-1}(G)$$

is a G_δ-subset of $\mathcal{I}$.

As pointed out by the authors of [2], the proofs of several papers on this topic just reproduce the proof of Baire's Theorem. Here we use Baire's Theorem and deduce a number of consequences related with Liouville numbers in the subsequent sections.

In Sect. 3, we deduce corollaries from Proposition 1.

In Sect. 4, we deduce from Proposition 2 some results of algebraic independence for Liouville numbers related to Schanuel's Conjecture.

2. Preliminaries. The following Proposition generalizes Proposition 1. We replace the interval $\mathcal{I}$ by a topological space $\mathbf{X}$, and we replace $\mathbf{R}$ by an interval $\mathcal{J}$.

Proposition 2. *Let $\mathbf{X}$ be a complete, locally connected topological space, $\mathcal{J}$ an interval in $\mathbf{R}$, and $\mathcal{N}$ a set which is either finite or else countable. For each $n \in \mathcal{N}$, let G_n be a G_δ-subset of $\mathcal{J}$ and let $f_n : \mathbf{X} \to \mathcal{J}$ be a continuous function which is nowhere locally constant. Then $\bigcap_{n\in\mathcal{N}} f_n^{-1}(G_n)$ is a G_δ-subset of $\mathbf{X}$.*

Proof. Since $\mathcal{N}$ is at most countable, it is enough to prove for any $n \in \mathcal{N}$ that $f_n^{-1}(G_n)$ is a G_δ-subset of $\mathbf{X}$.

Since f_n is continuous, $f_n^{-1}(G_n)$ is a countable intersection of open sets in $\mathbf{X}$. To prove it is a G_δ-subset of $\mathbf{X}$, we need to prove that $f_n^{-1}(G_n)$ is dense in $\mathbf{X}$, using the assumption that f_n is nowhere locally constant. Let V be a connected open subset of $\mathbf{X}$. Since f_n is continuous, $f_n(V)$ is a connected subset of $\mathcal{J}$. Since f_n is nowhere locally constant, $f_n(V)$ consists of at least two elements. Therefore, there exists an interval $(a, b) \subset \mathcal{J}$ with non-empty interior such that $(a, b) \subset f_n(V)$. Since G_n is a dense subset of $\mathcal{J}$, we have $(a, b) \cap G_n \neq \emptyset$ and hence $V \cap f_n^{-1}(G_n) \neq \emptyset$, which proves Proposition 2. □

We close this section with the following lemmas and a corollary (quoted in [2]).

Lemma 1. *Let $\mathbf{X}$ be a (nonempty) complete metric space without isolated point, and let E be a G_δ-subset of $\mathbf{X}$. Let F be a countable subset of E. Then $E\backslash F$ is a G_δ-subset of $\mathbf{X}$.*

Proof. We have

$$E\backslash F = \bigcap_{y\in F} E\backslash\{y\},$$

where each $E\backslash\{y\}$ is a G_δ-subset of $\mathbf{X}$ (since $\mathbf{X}$ has no isolated point). □

Using Baire's theorem, we deduce the following corollary.

Corollary 1. *Let $\mathbf{X}$ be a (nonempty) complete metric space without isolated point, and let E be a G_δ-subset of $\mathbf{X}$. Then E is uncountable.*

The next auxiliary lemma will be useful in Sect. 3 (proof of Corollary 5).

Lemma 2. *Let $\mathcal{I}_1, \ldots, \mathcal{I}_n$ be non-empty open subsets of* $\mathbf{R}$. *For each $i = 1, \ldots, n$, let G_i be a G_δ-subset of $\mathcal{I}_i$. Then there exists $(\xi_1, \ldots, \xi_n) \in G_1 \times \cdots \times G_n$ such that $\xi_1, \ldots, \xi_n$ are algebraically independent (over* $\mathbf{Q}$*).*

Proof. We prove Lemma 2 by induction on n. For $n = 1$, it follows from Corollary 1 that the intersection of G_1 with the set of transcendental numbers is not empty.

Assume Lemma 2 holds for $n-1$. There exists $(\xi_1, \ldots, \xi_{n-1}) \in G_1 \times \cdots \times G_{n-1}$ such that $\xi_1, \ldots, \xi_{n-1}$ are algebraically independent. The set of $\xi_n \in \mathcal{I}_n$ which are transcendental over $\mathbf{Q}(\xi_1, \ldots, \xi_{n-1})$ is a G_δ-subset of $\mathcal{I}_n$, hence its intersection with G_n is again a G_δ-subset: it is dense by Baire's theorem, and therefore not empty. □

3. Application of Proposition 1 to Liouville numbers. Since the set of Liouville numbers is a G_δ-subset in $\mathbf{R}$, a direct consequence of Proposition 1 and Corollary 1 is the following:

Corollary 2. *Let $\mathcal{I}$ be an interval of* $\mathbf{R}$ *with nonempty interior and $(f_n)_{n\geq 1}$ a sequence of real maps on $\mathcal{I}$ which are continuous and nowhere locally constant. Then there exists an uncountable subset E of $\mathcal{I} \cap \mathbb{L}$ such that $f_n(\xi)$ is a Liouville number for all $n \geq 1$ and all $\xi \in E$.*

Define $f_0 : \mathcal{I} \to \mathbf{R}$ as the identity $f_0(x) = x$. The conclusion of the Corollary 2 is that

$$E = \bigcap_{n\geq 0} f_n^{-1}(\mathbb{L}),$$

a subset of $\mathcal{I}$, is uncountable.

In this section we deduce consequences of Corollary 2. We first consider the special case, where all f_n are the same.

Corollary 3. *Let $\mathcal{I}$ be an interval of* $\mathbf{R}$ *with nonempty interior and $\varphi : \mathcal{I} \to \mathbf{R}$ a continuous map which is nowhere locally constant. Then there exists an uncountable set of Liouville numbers $\xi \in \mathcal{I}$ such that $\varphi(\xi)$ is a Liouville number.*

One can deduce Corollary 3 directly from Proposition 1 by taking all $f_n = \varphi$ $(n \geq 0)$ and $G = \mathbb{L}$ and by noticing that the intersection of the two G_δ-subsets $\varphi^{-1}(\mathbb{L})$ and $\mathbb{L}$ is uncountable. Another proof is to use Proposition 1 with $f_0(x) = x$ and $f_n(x) = \varphi(x)$ for $n \geq 1$ and $G = \mathbb{L}$.

Simple examples of consequences of Corollary 3 are obtained with $\mathcal{I} = (0, +\infty)$ and either $\varphi(x) = t - x$, for $t \in \mathbf{R}$, or else with $\varphi(x) = t/x$, for $t \in \mathbf{R}^\times$, which yield Erdős above mentioned result on the decomposition of any real number (resp. any nonzero real number) t as a sum (resp. a product) of two Liouville numbers.

We deduce also from Corollary 3 that any positive real number t is the sum of two squares of Liouville numbers. This follows by applying Corollary 3 with

$$\mathcal{I} = (0, \sqrt{t}) \quad \text{and} \quad \varphi(x) = \sqrt{t - x^2}.$$

Similar examples can be derived from Corollary 3 involving transcendental functions: for instance, any real number can be written $e^\xi + \eta$ with ξ and η

Liouville numbers; any positive real number can be written $e^{\xi} + e^{\eta}$ with ξ and η Liouville numbers.

Using the implicit function theorem, one deduces from Corollary 3 the following generalization of Erdős's result.

Corollary 4. *Let $P \in \mathbf{R}[X, Y]$ be an irreducible polynomial such that $(\partial/\partial X)P \neq 0$ and $(\partial/\partial Y)P \neq 0$. Assume that there exist two nonempty open intervals $\mathcal{I}$ and $\mathcal{J}$ of $\mathbf{R}$ such that, for any $x \in \mathcal{I}$, there exists $y \in \mathcal{J}$ with $P(x, y) = 0$, and, for any $y \in \mathcal{J}$, there exists $x \in \mathcal{I}$ with $P(x, y) = 0$. Then there exist uncountably many pairs (ξ, η) of Liouville numbers in $\mathcal{I} \times \mathcal{J}$ such that $P(\xi, \eta) = 0$.*

Proof of Corollary 4. We use the implicit function Theorem (for instance Theorem 2 of [4]) to deduce that there exist two differentiable functions φ and ψ, defined on nonempty open subsets $\mathcal{I}'$ of $\mathcal{I}$ and $\mathcal{J}'$ of $\mathcal{J}$, such that $P\big(x, \varphi(x)\big) = 0$ and $P\big(\psi(y), y\big) = 0$ for $x \in \mathcal{I}'$ and $y \in \mathcal{J}'$, and such that $\varphi \circ \psi$ is the identity on $\mathcal{J}'$ and $\psi \circ \varphi$ is the identity on $\mathcal{I}'$. We then apply Corollary 3. □

Erdős's result on $t = \xi + \eta$ for $t \in \mathbf{R}$ follows from Corollary 4 with $P(X, Y) = X + Y - t$, while his result on $t = \xi\eta$ for $t \in \mathbf{R}^{\times}$ follows with $P(X, Y) = XY - t$. Also, the above mentioned fact that any positive real number t is the sum of two squares of Liouville numbers follows by applying Corollary 4 to the polynomial $X^2 + Y^2 - t$.

One could also deduce, under the hypotheses of Corollary 4, the existence of one pair of Liouville numbers (ξ, η) with $P(\xi, \eta) = 0$ by applying Theorem 1 of [4] with $f(x, y) = P(x, y)$ and $\alpha = 0$. The proof we gave produces an uncountable set of solutions.

We extend Corollary 4 to more than 2 variables as follows:

Corollary 5. *Let $\ell \geq 2$, and let $P \in \mathbf{R}[X_1, \dots, X_\ell]$ be an irreducible polynomial such that $(\partial/\partial X_1)P \neq 0$ and $(\partial/\partial X_2)P \neq 0$. Assume that there exist nonempty open subsets $\mathcal{I}_i$ of $\mathbf{R}$ $(i = 1, \dots, \ell)$ such that, for any $i \in \{1, 2\}$ and any $(\ell - 1)$–tuple $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_\ell) \in \mathcal{I}_1 \times \cdots \times \mathcal{I}_{i-1} \times \mathcal{I}_{i+1} \times \cdots \times \mathcal{I}_\ell$, there exists $x_i \in \mathcal{I}_i$ such that $P(x_1, \dots, x_\ell) = 0$. Then there exist uncountably many tuples $(\xi_1, \xi_2, \dots, \xi_\ell) \in \mathcal{I}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_\ell$ of Liouville numbers such that $P(\xi_1, \xi_2, \dots, \xi_\ell) = 0$.*

Proof. For $\ell = 2$, this is Corollary 4. Assume $\ell \geq 3$. Using Lemma 2, we know that there exists a $(\ell - 2)$–tuple of $\mathbf{Q}$–algebraically independent Liouville numbers $(\xi_3, \dots, \xi_\ell)$ in $\mathcal{I}_3 \times \cdots \times \mathcal{I}_\ell$. We finally apply Corollary 4 to the polynomial $P(X_1, X_2, \xi_3, \dots, \xi_\ell) \in \mathbf{R}[X_1, X_2]$. □

In [5], using a counting argument together with an application of Bézout's Theorem, Burger proved that *an irrational number t is transcendental if and only if there exist two $\mathbf{Q}$–algebraically independent Liouville numbers ξ and η such that $t = \xi + \eta$.* Extending the method of [5], we prove:

Proposition 3. *Let $F(X, Y) \in \mathbf{Q}[X, Y]$ be a nonconstant polynomial with rational coefficients and t a real number. Assume that there is an uncountable* *set of pairs of Liouville numbers* (ξ, η) *such that* $F(\xi, \eta) = t$. *Then the two following conditions are equivalent.*

(i) t *is transcendental.*

(ii) *there exist two* $\mathbf{Q}$*–algebraically independent Liouville numbers* (ξ, η) *such that* $F(\xi, \eta) = t$.

Proof of Proposition 3. Assume t is algebraic. Therefore there exists $P(X) \in \mathbf{Q}[X]\backslash\{0\}$ such that $P(t) = 0$. For any pair of Liouville numbers (ξ, η) such that $F(\xi, \eta) = t$, we have $P(F(\xi, \eta)) = 0$. Since $P \circ F \in \mathbf{Q}[X, Y]\backslash\{0\}$, we deduce that the numbers ξ and η are algebraically dependent.

Conversely, assume that for any pair of Liouville numbers (ξ, η) such that $F(\xi, \eta) = t$, the numbers ξ and η are algebraically dependent. Since $\mathbf{Q}[X, Y]$ is countable and since there is an uncountable set of such pairs of Liouville numbers (ξ, η), there exists a nonzero polynomial $A \in \mathbf{Q}[X, Y]$ such that $A(X, Y)$ and $F(X, Y) - t$ have infinitely many common zeros (ξ, η). We use Bézout's Theorem. We decompose $A(X, Y)$ into irreducible factors in $\overline{\mathbf{Q}}[X, Y]$, where $\overline{\mathbf{Q}}$ is the algebraic closure of $\mathbf{Q}$. One of these factors, say $B(X, Y)$, divides $F(X, Y) - t$ in $\overline{\mathbf{Q}(t)}[X, Y]$, where $\overline{\mathbf{Q}(t)}$ denotes the algebraic closure of $\mathbf{Q}(t)$.

Assume now that t is transcendental. Write $F(X, Y) - t = B(X, Y)C(X, Y)$, where $C \in \overline{\mathbf{Q}(t)}[X, Y]$. The coefficient of a monomial X^iY^j in C is

$$\left(\frac{\partial^{i+j}}{\partial X^i \partial Y^j}\right)\left(\frac{F(X, Y) - t}{B(X, Y)}\right)(0, 0),$$

hence $C \in \overline{\mathbf{Q}}[t, X, Y]$ and C has degree 1 in t, say $C(X, Y) = D(X, Y) + tE(X, Y)$, with D and E in $\overline{\mathbf{Q}}[t, X, Y]$. Therefore $B(X, Y)E(X, Y) = -1$, contradicting the fact that $B(X, Y)$ is irreducible. □

Here is a consequence of Corollary 2, where a sequence of (f_n) is involved, not only one φ like in Corollary 3.

Corollary 6. *Let* $\mathcal{E}$ *be a countable subset of* $\mathbf{R}$. *Then there exists an uncountable set of positive Liouville numbers* ξ *having simultaneously the following properties.*

(i) *For any* $t \in \mathcal{E}$, *the number* $\xi + t$ *is a Liouville number.*

(ii) *For any nonzero* $t \in \mathcal{E}$, *the number* ξt *is a Liouville number.*

(iii) *Let* $t \in \mathcal{E}$, $t \neq 0$. *Define inductively* $\xi_0 = \xi$ *and* $\xi_n = e^{t\xi_{n-1}}$ *for* $n \geq 1$. *Then all numbers of the sequence* $(\xi_n)_{n\geq 0}$ *are Liouville numbers.*

(iv) *For any rational number* $r \neq 0$, *the number* ξ^r *is a Liouville number.*

Proof of Corollary 6. Each of the four following sets of continuous real maps defined on $\mathcal{I} = (0, +\infty)$ is countable, hence their union is countable. We enumerate the elements of the union, and we apply Corollary 2.

The first set consists of the maps $x \mapsto x + t$ for $t \in \mathcal{E}$.

The second set consists of the maps $x \mapsto xt$ for $t \in \mathcal{E}$, $t \neq 0$.

The third set consists of the maps φ_n defined inductively by $\varphi_0(x) = x$, $\varphi_n(x) = e^{t\varphi_{n-1}(x)}$ $(n \geq 1)$.

The fourth set consists of the maps $\varphi_r(x) = x^r$ for any rational number $r \neq 0$. □

In [8], Maillet gives a necessary and sufficient condition for a positive Liouville number ξ to have a p–th root (for a given positive integer $p > 1$) which is also a Liouville number: among the convergents in the continued fraction expansion of ξ, infinitely many should be p–th powers. He provides explicit examples of Liouville numbers having a p–th root which is not a Liouville number.

Let $\mathcal{I}$ be an interval of $\mathbf{R}$ with nonempty interior and $\varphi : \mathcal{I} \to \mathcal{I}$ a continuous bijective map (hence φ is nowhere locally constant). Let $\psi : \mathcal{I} \to \mathcal{I}$ denote the inverse bijective map of φ. For $n \in \mathbf{Z}$, we denote by φ^n the bijective map $\mathcal{I} \to \mathcal{I}$ defined inductively as usual: φ^0 is the identity, $\varphi^n = \varphi^{n-1} \circ \varphi$ for $n \geq 1$, and $\varphi^{-n} = \psi^n$ for $n \geq 1$.

Here is a further consequence of Corollary 2.

Corollary 7. *Let $\mathcal{I}$ be an interval of $\mathbf{R}$ with nonempty interior and $\varphi : \mathcal{I} \to \mathcal{I}$ a continuous bijective map. Then the set of elements ξ in $\mathcal{I}$ such that the orbit $\{\varphi^n(\xi) \mid n \in \mathbf{Z}\}$ consists only of Liouville numbers in $\mathcal{I}$ is a G_δ-subset of $\mathcal{I}$, hence is uncountable.*

Proof of Corollary 7. In Proposition 2, take $X = \mathcal{I}$, $\mathcal{N} = \mathbf{Z}$, $G_n = \mathbb{L} \cap \mathcal{I}$, and $f_n = \varphi^n$ for each $n \in \mathbf{Z}$. $\square$

4. Algebraic independence. Schanuel's Conjecture (see [12] § IV. 1) states that, given $\mathbf{Q}$–linearly independent complex numbers $x_1, \ldots, x_n$, the transcendence degree over $\mathbf{Q}$ of the field

$$\mathbf{Q}(x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}) \tag{1}$$

is at least n. One may ask whether the transcendence degree is at least $n+1$ when the following additional assumption is made: for each $i = 1, \ldots, n$, one at least of the two numbers x_i, e^{x_i} is a Liouville number.

We will show that for each pair of integers (n, m) with $n \geq m \geq 1$, there exist uncountably many tuples $\xi_1, \ldots, \xi_n$ consisting of $\mathbf{Q}$–linearly independent real numbers, such that the numbers $\xi_1, \ldots, \xi_n, e^{\xi_1}, \ldots, e^{\xi_n}$ are all Liouville numbers, and the transcendence degree of the field (1) is $n + m$.

For L a field and K a subfield of L, we denote by $\mathrm{trdeg}_K L$ the transcendence degree of L over K.

Theorem 1. *Let $n \geq 1$ and $1 \leq m \leq n$ be given integers. Then there exist uncountably many n-tuples $(\alpha_1, \ldots, \alpha_n) \in \mathbb{L}^n$ such that $\alpha_1, \ldots, \alpha_n$ are linearly independent over $\mathbf{Q}$, $e^{\alpha_i} \in \mathbb{L}$ for all $i = 1, 2, \ldots, n$ and*

$$\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(\alpha_1, \ldots, \alpha_n, e^{\alpha_1}, \ldots, e^{\alpha_n}) = n + m.$$

Remark. *Theorem 1 is tight when $n = 1$: the result does not hold for $m = 0$. Indeed, since the set of α in $\mathbb{L}$ such that α and e^α are algebraically dependent over $\mathbf{Q}$ is countable, one cannot get uncountably many $\alpha \in \mathbb{L}$ such that* $\mathrm{trdeg}_{\mathbf{Q}} \mathbf{Q}(\alpha, e^\alpha) = 1$.

We need an auxiliary result (Corollary 8). Corollary 8 will be deduced from the following Proposition 4.

Proposition 4. (1) *Let $g_1, g_2, \ldots, g_n$ be polynomials in $\mathbf{C}[z]$. Then the two following conditions are equivalent.*

   (i) *For* $1 \le i < j \le n$, *the function* $g_i - g_j$ *is not constant.*
   (ii) *The functions* $e^{g_1}, \dots, e^{g_n}$ *are linearly independent over* $\mathbf{C}(z)$.

(2) *Let* $f_1, f_2, \dots, f_m$ *be polynomials in* $\mathbf{C}[z]$. *Then the two following conditions are equivalent.*
   (i) *For any nonzero tuple* $(a_1, \dots, a_m) \in \mathbf{Z}^m$, *the function* $a_1 f_1 + \cdots + a_m f_m$ *is not constant.*
   (ii) *The functions* $e^{f_1}, \dots, e^{f_m}$ *are algebraically independent over* $\mathbf{C}(z)$.

Since the functions $1, z, z^2, \dots, z^m, \dots$ are linearly independent over $\mathbf{C}$, we deduce from (2):

Corollary 8. *The functions*

$$z,\ e^z,\ e^{z^2},\ \dots,\ e^{z^m}, \dots$$

are algebraically independent over $\mathbf{C}$.

For the proof of Proposition 4, we introduce the quotient vector space $\mathcal{V} = \mathbf{C}[z]/\mathbf{C}$ and the canonical surjective linear map $s : \mathbf{C}[z] \to \mathcal{V}$ with kernel $\mathbf{C}$. Assertion (i) in (1) means that $s(g_1), \dots, s(g_n)$ are pairwise distinct, while assertion (i) in (2) means that $s(f_1), \dots, s(f_m)$ are linearly independent over $\mathbf{Q}$.

Proof of (1). (ii) implies (i). If $g_1 - g_2$ is a constant c, then (e^{g_1}, e^{g_2}) is a zero of the linear form $X_1 - e^c X_2$.

(i) implies (ii). We prove this result by induction on n. For $n = 1$, there is no condition on g_1, the function e^{g_1} is not zero, hence the result is true. Assume $n \ge 2$ and assume that for any $n' < n$, the result holds with n replaced by n'. Let $A_1, \dots, A_n$ be polynomials in $\mathbf{C}[z]$, not all of which are zero; consider the function

$$G(z) = A_1(z)e^{g_1(z)} + \cdots + A_n(z)e^{g_n(z)}.$$

The goal is to check that G is not the zero function. Using the induction hypothesis, we may assume $A_i \ne 0$ for $1 \le i \le n$. Define $h_i = g_i - g_n$ $(1 \le i \le n)$ and $H = e^{-g_n} G$:

$$H(z) = A_1(z)e^{h_1(z)} + \cdots + A_{n-1}(z)e^{h_{n-1}(z)} + A_n(z).$$

From $h_i - h_j = g_i - g_j$, we deduce that $s(h_1), \dots, s(h_{n-1})$ are distinct in $\mathcal{V}$. Write $D = d/dz$, and let $N > \deg A_n$, so that $D^N A_n = 0$. Notice that for $i = 1, \dots, n-1$ and for $t \ge 0$, we can write

$$D^t\big(A_i(z)e^{h_i(z)}\big) = A_{it}(z)e^{h_i(z)},$$

where A_{it} is a nonzero polynomial in $\mathbf{C}[z]$. By the induction hypothesis, the function

$$D^N H(z) = A_{1,N}(z)e^{h_1(z)} + \cdots + A_{n-1,N}(z)e^{h_{n-1}(z)}$$

is not the zero function, hence $G \ne 0$. □

Proof of (2). (*ii*) implies (*i*). If there exists $(a_1,\dots,a_m)\in \mathbf{Z}^m\backslash\{(0,\dots,0)\}$ such that the function $a_1f_1+\cdots+a_mf_m$ is a constant c, then for the polynomial

$$P(X_1,\dots,X_m)=\prod_{a_i>0}X_i^{a_i}-e^c\prod_{a_i<0}X_i^{|a_i|},$$

we have $P(e^{f_1},\dots,e^{f_m})=0$, therefore the functions $e^{f_1},\dots,e^{f_m}$ are algebraically dependent over $\mathbf{C}$ (hence over $\mathbf{C}(z)$).

(*i*) implies (*ii*). Consider a nonzero polynomial

$$P(X_1,\dots,X_m)=\sum_{\lambda_1=0}^{d_1}\cdots\sum_{\lambda_m=0}^{d_m}p_{\lambda_1,\dots,\lambda_m}(z)X_1^{\lambda_1}\cdots X_m^{\lambda_m}\in\mathbf{C}[z,X_1,\dots,X_m],$$

and let F be the entire function $F=P(e^{f_1},\dots,e^{f_m})$. Denote by $\{g_1,\dots,g_n\}$ the set of functions $\lambda_1f_1+\cdots+\lambda_mf_m$ with $p_{\lambda_1,\dots,\lambda_m}\neq 0$. For $1\le i\le n$, set

$$A_i(z)=p_{\lambda_1,\dots,\lambda_m}(z)\in\mathbf{C}[z],$$

where $(\lambda_1,\dots,\lambda_m)$ is defined by $g_i=\lambda_1f_1+\cdots+\lambda_mf_m$, so that

$$F(z)=A_1(z)e^{g_1(z)}+\cdots+A_n(z)e^{g_n(z)}.$$

The assumption (*i*) of (2) on $f_1,\dots,f_m$ implies that the functions $g_1,\dots,g_n$ satisfy the assumption (*i*) of (1), hence the function F is not the zero function. □

Remark. *We deduced* (2) *from* (1)*. We can also deduce* (1) *from* (2) *as follows. Assume that* (*ii*) *in* (1) *is not true, meaning that the functions* $e^{g_1},\dots,e^{g_n}$ *are linearly dependent over* $\mathbf{C}(z)$*: there exist polynomials* $A_1,\dots,A_n$*, not all of which are zero, such that the function*

$$G(z)=A_1(z)e^{g_1(z)}+\cdots+A_n(z)e^{g_n(z)}$$

is the zero function. Consider a set $f_1,\dots,f_m$ *of polynomials such that* $s(f_1),\dots,s(f_m)$ *is a basis of the* $\mathbf{Q}$*–vector subspace of* $\mathcal{V}$ *spanned by* $s(g_1),\dots,s(g_n)$*. Dividing if necessary all* f_j *by a positive integer, we may assume*

$$s(g_i)=\sum_{j=1}^m\lambda_{ij}s(f_j)\quad(1\le i\le n)$$

with $\lambda_{ij}\in\mathbf{Z}$*. This means that*

$$c_i=g_i-\sum_{j=1}^m\lambda_{ij}f_j\quad(1\le i\le n)$$

are constants. Consider the polynomial

$$P(X_1,\dots,X_m)=\sum_{i=1}^nA_i(z)e^{c_i}\prod_{j=1}^mX_j^{\lambda_{ij}}.$$

From $P(e^{f_1},\ldots,e^{f_m}) = G = 0$ *and from (2), we deduce that this polynomial is* 0*, hence the monomials*

$$\prod_{j=1}^{m} X_j^{\lambda_{ij}} \quad (1 \le i \le n)$$

are not pairwise distinct: there exists $i_1 \ne i_2$ *with*

$$\lambda_{i_1 j} = \lambda_{i_2 j} \quad \textit{for } 1 \le j \le m.$$

Therefore $g_{i_1} - g_{i_2} = c_{i_1} - c_{i_2}$*, hence* $s(g_{i_1}) = s(g_{i_2})$*, which means that* (*i*) *in* (1) *is not true.*

Proof of Theorem 1. Let n and m be integers such that $1 \le m \le n$. We shall prove the assertion by induction on $m \ge 1$. Assume $m = 1$. We prove the result for all $n \ge 1$. For each nonzero polynomial $P(X_0, X_1, \ldots, X_n) \in \mathbf{Q}[X_0,\ldots,X_n]$ in $n+1$ variables with rational coefficients, define a function

$$f_P : \mathbf{R} \to \mathbf{R} \text{ by } f_P(x) = P(x, e^x, \ldots, e^{x^n}).$$

Using Corollary 8, we deduce that the set $Z(f_P)$ of all real zeros of f_p, as the real zero locus of a non-zero complex analytic map f_P, is discrete in $\mathbf{R}$, hence that its complement is open and dense in $\mathbf{R}$. From Proposition 1 and Baire's theorem, it follows that the set

$$E = \left\{\alpha \in \mathbb{L} \mid \quad e^{\alpha^j} \in \mathbb{L} \text{ for } j = 1,\ldots,n\right\} \cap \bigcap_{P \in \mathbf{Q}[X_0,\ldots,X_n]\setminus\{0\}} (\mathbf{R} \setminus Z(f_P))$$

is a G_δ-subset of $\mathbf{R}$. Therefore, by Corollary 1 , E is uncountable. For any $\alpha \in E$, the numbers $\alpha, e^\alpha, e^{\alpha^2}, \ldots, e^{\alpha^n}$ are in $\mathbb{L}$ and are algebraically independent over $\mathbf{Q}$. Since α is a Liouville number, $\alpha^2, \ldots, \alpha^n$ are also Liouville numbers, and $\alpha, \alpha^2, \ldots, \alpha^n$ are linearly independent over $\mathbf{Q}$. From

$$\operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(\alpha, \alpha^2, \ldots, \alpha^n, e^\alpha, \ldots, e^{\alpha^n}) = n+1$$

we conclude that the assertion is true for $m = 1$ and for all $n \ge 1$.

Assume that $1 < m \le n$. Also, suppose the assertion is true for $m-1$ and for all $n \ge m-1$. In particular, the assertion is true for $m-1$ and $n-1$. Hence, there are uncountably many $n-1$ tuples $(\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{L}^{n-1}$ such that $\alpha_1, \ldots, \alpha_{n-1}$ are linearly independent over $\mathbf{Q}$, $e^{\alpha_1}, \ldots, e^{\alpha_{n-1}}$ are Liouville numbers, and

$$\operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(\alpha_1, \ldots, \alpha_{n-1}, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}}) = n+m-2. \tag{2}$$

Choose such an $(n-1)$-tuple $(\alpha_1, \ldots, \alpha_{n-1})$. Consider the subset E of $\mathbf{R}$ which consists of all $\alpha \in \mathbf{R}$ such that α, e^α are algebraically independent over $\mathbf{Q}(\alpha_1, \ldots \alpha_{n-1}, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}})$.

If $P(X,Y) \in \mathbf{Q}(\alpha_1, \ldots, \alpha_{n-1}, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}})[X,Y]$ is a polynomial, define an analytic function $f(z) = P(z, e^z)$ in $\mathbf{C}$. Since z, e^z are algebraically independent functions over $\mathbf{C}$ (by Corollary 8), if P is a nonzero polynomial, f is a nonzero function. Therefore, the set of zeros of f in $\mathbf{C}$ is countable. Since there are only countably many polynomials $P(X,Y)$ with coefficients in the field $\mathbf{Q}(\alpha_1, \ldots \alpha_{n-1}, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}})$, we conclude that $\mathbf{R} \setminus E$ is countable. Therefore $F = E \cap \mathbb{L}$ is uncountable. For each $\alpha \in F$, the two numbers α, e^α

are algebraically independent over $\mathbf{Q}(\alpha_1, \ldots, \alpha_{n-1}, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}})$. From (2) we deduce

$$\operatorname{trdeg}_{\mathbf{Q}} \mathbf{Q}(\alpha_1, \ldots, \alpha_{n-1}, \alpha, e^{\alpha_1}, \ldots, e^{\alpha_{n-1}}, e^{\alpha}) = n + m.$$

This completes the proof of Theorem 1. □

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References

[1] K. Alniaçik, Representation of real numbers as sum of U_2 numbers, Acta Arith. **55** (1990), 301–310.

[2] K. Alniaçik and É. Saias, Une remarque sur les G_δ–denses, Arch. Math. (Basel) **62** (1994), 425–426.

[3] Y. Bugeaud, Approximation by algebraic numbers. Cambridge Tracts in Mathematics, **160**. Cambridge University Press, Cambridge, 2004.

[4] E. B. Burger, On Liouville decompositions in local fields, Proc. Amer. Math. Soc, **124** (1996), 3305–3310.

[5] E. B. Burger, Diophantine inequalities and irrationality measures for certain transcendental numbers, Indian J. Pure Appl. Math. **32** (2001), 1591–1599.

[6] P. Erdős, Representation of real numbers as sums and products of Liouville numbers, Michigan Math. J, **9** (1) (1962) 59–60.

[7] J. Liouville, Sur des classes très étendues de quantités dont la valeur n'est ni algébrique, ni même réductible à des irrationnelles algébriques, J. Math. Pures et Appl. **18** (1844) 883–885, and 910–911.

[8] É. Maillet, Introduction à la théorie des nombres transcendants et des propriétés arithmétiques des fonctions, Paris, 1906.

[9] É. Maillet, Sur quelques propriétés des nombres transcendants de Liouville, Bull. S.M.F **50** (1922), 74–99.

[10] G. J. Rieger, Über die Lösbarkeit von Gleichungssystemen durch Liouville–Zahlen. (1975), 40–43.

[11] W. Schwarz, Liouville–Zahlen und der Satz von Baire, Math.-Phys. Semesterber. **24** (1977), 84–87.

[12] M. Waldschmidt, Algebraic independence of transcendental numbers. In: Gel'fond's method and its developments. *Perspectives in mathematics,* 551–571, Birkhäuser, Basel-Boston, Mass., 1984.

[13] M. Waldschmidt, Indépendance algébrique de nombres de Liouville, Cinquante ans de polynômes, Lecture Notes in Math. (Springer Verlag), **1415** (1990), 225–235.

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