PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 143, Number 8, August 2015, Pages 3215–3229
http://dx.doi.org/10.1090/proc/12408
Article electronically published on April 22, 2015

LIOUVILLE NUMBERS, LIOUVILLE SETS AND LIOUVILLE FIELDS

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(Communicated by Matthew A. Papanikolas)

Abstract. Following earlier work by É. Maillet 100 years ago, we introduce the definition of a *Liouville set*, which extends the definition of a Liouville number. We also define a *Liouville field*, which is a field generated by a Liouville set. Any Liouville number belongs to a Liouville set S having the power of continuum and such that $\mathbf{Q} \cup \mathsf{S}$ is a Liouville field.

1. Introduction

For any integer q and any real number $x \in \mathbf{R}$, we denote by

$$\|qx\| = \min_{m \in \mathbf{Z}} |qx - m|$$

the distance of qx to the nearest integer. Following É. Maillet [3, 4], an irrational real number ξ is said to be a *Liouville number* if, for each integer $n \geq 1$, there exists an integer $q_n \geq 2$ such that the sequence $\big(u_n(\xi)\big)_{n\geq 1}$ of real numbers defined by

$$u_n(\xi) = -\frac{\log \|q_n \xi\|}{\log q_n}$$

satisfies $\lim_{n\to\infty} u_n(\xi) = \infty$. If p_n is the integer such that $\|q_n\xi\| = |\xi q_n - p_n|$, then the definition of $u_n(\xi)$ can be written:

$$|q_n\xi - p_n| = \frac{1}{q_n^{u_n(\xi)}}\cdot$$

An equivalent definition is to say that a Liouville number is a real number ξ such that, for each integer $n \geq 1$, there exists a rational number p_n/q_n with $q_n \geq 2$ such that

$$0 < \left|\xi - \frac{p_n}{q_n}\right| \leq \frac{1}{q_n^n}\cdot$$

We denote by $\mathbb{L}$ the set of Liouville numbers. Following [2], any Liouville number is a transcendental number.

We introduce the notions of a *Liouville set* and a *Liouville field*. They extend what was done by É. Maillet in Chap. III of [3].

Received by the editors May 27, 2013.
2010 *Mathematics Subject Classification.* Primary 11J82.
Key words and phrases. Liouville number, Liouville set, Liouville field, continuum, G_δ-subsets.

Definition. A *Liouville set* is a subset S of $\mathbb{L}$ for which there exists an increasing sequence $(q_n)_{n\geq 1}$ of positive integers having the following property: for any $\xi \in \mathsf{S}$, there exists a sequence $\big(b_n\big)_{n\geq 1}$ of positive rational integers and there exist two positive constants κ_1 and κ_2 such that, for any sufficiently large n,

$$(1) \qquad 1 \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 n}}\cdot$$

It would not make a difference if we were requesting these inequalities to hold for any $n \geq 1$: it suffices to change the constants κ_1 and κ_2.

Definition. A *Liouville field* is a field of the form $\mathbf{Q}(\mathsf{S})$ where S is a Liouville set.

From the definitions, it follows that, for a real number ξ, the following conditions are equivalent:
(i) ξ is a Liouville number.
(ii) ξ belongs to some Liouville set.
(iii) The set $\{\xi\}$ is a Liouville set.
(iv) The field $\mathbf{Q}(\xi)$ is a Liouville field.

If we agree that the empty set is a Liouville set and that $\mathbf{Q}$ is a Liouville field, then any subset of a Liouville set is a Liouville set, and also (see Theorem 1) any subfield of a Liouville field is a Liouville field.

Definition. Let $\underline{q} = (q_n)_{n\geq 1}$ be an increasing sequence of positive integers and let $\underline{u} = (u_n)_{n\geq 1}$ be a sequence of positive real numbers such that $u_n \to \infty$ as $n \to \infty$. Denote by $\mathsf{S}_{\underline{q},\underline{u}}$ the set of $\xi \in \mathbb{L}$ such that there exist two positive constants κ_1 and κ_2 and there exists a sequence $\big(b_n\big)_{n\geq 1}$ of positive rational integers with

$$1 \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 u_n}}\cdot$$

Denote by $\underline{n}$ the sequence $\underline{u} = (u_n)_{n\geq 1} := (1, 2, 3, \dots)$ with $u_n = n$ $(n \geq 1)$. For any increasing sequence $\underline{q} = (q_n)_{n\geq 1}$ of positive integers, we denote by $\mathsf{S}_{\underline{q}}$ the set $\mathsf{S}_{\underline{q},\underline{n}}$.

Hence, by definition, a Liouville set is a subset of some $\mathsf{S}_{\underline{q}}$. In section 2 we prove the following lemma:

Lemma 1. *For any increasing sequence $\underline{q}$ of positive integers and any sequence $\underline{u}$ of positive real numbers which tends to infinity, the set $\mathsf{S}_{\underline{q},\underline{u}}$ is a Liouville set.*

Notice that if $(m_n)_{n\geq 1}$ is an increasing sequence of positive integers, then for the subsequence $\underline{q}' = (q_{m_n})_{n\geq 1}$ of the sequence $\underline{q}$, we have $\mathsf{S}_{\underline{q}',\underline{u}} \supset \mathsf{S}_{\underline{q},\underline{u}}$.

Example. Let $\underline{u} = (u_n)_{n\geq 1}$ be a sequence of positive real numbers which tends to infinity. Define $f : \mathbf{N} \to \mathbf{R}_{>0}$ by $f(1) = 1$ and

$$f(n) = u_1 u_2 \cdots u_{n-1} \qquad (n \geq 2),$$

so that $f(n+1)/f(n) = u_n$ for $n \geq 1$. Define the sequence $\underline{q} = (q_n)_{n\geq 1}$ by $q_n = \lfloor 2^{f(n)} \rfloor$. Then, for any real number $t > 1$, the number

$$\xi_t = \sum_{n\geq 1} \frac{1}{\lfloor t^{f(n)} \rfloor}$$

belongs to $\mathsf{S}_{\underline{q},\underline{u}}$. The set $\{\xi_t \mid t > 1\}$ has the power of continuum, since $\xi_{t_1} < \xi_{t_2}$ for $t_1 > t_2 > 1$.

The sets $\mathsf{S}_{\underline{q},\underline{u}}$ have the following property (compare with Theorem I_3 in [3]):

Theorem 1. *For any increasing sequence $\underline{q}$ of positive integers and any sequence $\underline{u}$ of positive real numbers which tends to infinity, the set $\mathbf{Q}\cup\mathsf{S}_{\underline{q},\underline{u}}$ is a field.*

We denote this field by $\mathsf{K}_{\underline{q},\underline{u}}$, and by $\mathsf{K}_{\underline{q}}$ for the sequence $\underline{u}=\underline{n}$. From Theorem 1, it follows that a field is a Liouville field if and only if it is a subfield of some $\mathsf{K}_{\underline{q}}$. Another consequence is that, if S is a Liouville set, then $\mathbf{Q}(\mathsf{S})\setminus\mathbf{Q}$ is a Liouville set.

It is easily checked that if

$$\liminf_{n\to\infty}\frac{u_n}{u'_n}>0,$$

then $\mathsf{K}_{\underline{q},\underline{u}}$ is a subfield of $\mathsf{K}_{\underline{q},\underline{u}'}$. In particular if

$$\liminf_{n\to\infty}\frac{u_n}{n}>0,$$

then $\mathsf{K}_{\underline{q},\underline{u}}$ is a subfield of $\mathsf{K}_{\underline{q}}$, while if

$$\limsup_{n\to\infty}\frac{u_n}{n}<+\infty,$$

then $\mathsf{K}_{\underline{q}}$ is a subfield of $\mathsf{K}_{\underline{q},\underline{u}}$.

If $R\in\mathbf{Q}(X_1,\dots,X_\ell)$ is a rational fraction and if $\xi_1,\dots,\xi_\ell$ are elements of a Liouville set S such that $\eta=R(\xi_1,\dots,\xi_\ell)$ is defined, then Theorem 1 implies that η is either a rational number or a Liouville number, and in the second case $\mathsf{S}\cup\{\eta\}$ is a Liouville set. For instance, if, in addition, R is not constant and $\xi_1,\dots,\xi_\ell$ are algebraically independent over $\mathbf{Q}$, then η is a Liouville number and $\mathsf{S}\cup\{\eta\}$ is a Liouville set. For $\ell=1$, this yields:

Corollary 1. *Let $R\in\mathbf{Q}(X)$ be a rational fraction and let ξ be a Liouville number. Then $R(\xi)$ is a Liouville number and $\{\xi,R(\xi)\}$ is a Liouville set.*

We now show that $\mathsf{S}_{\underline{q},\underline{u}}$ is either empty or else uncountable and we characterize such sets.

Theorem 2. *Let $\underline{q}$ be an increasing sequence of positive integers and $\underline{u}=(u_n)_{n\geq1}$ be an increasing sequence of positive real numbers such that $u_{n+1}\geq u_n+1$. Then the Liouville set $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty if and only if*

$$\limsup_{n\to\infty}\frac{\log q_{n+1}}{u_n\log q_n}>0.$$

Moreover, if the set $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty, then it has the power of continuum.

Let t be an irrational real number which is not a Liouville number. By a result due to P. Erdős [1], we can write $t=\xi+\eta$ with two Liouville numbers ξ and η. Let $\underline{q}$ be an increasing sequence of positive integers and $\underline{u}$ be an increasing sequence of real numbers such that $\xi\in\mathsf{S}_{\underline{q},\underline{u}}$. Since any irrational number in the field $K_{\underline{q},\underline{u}}$ is in $\mathsf{S}_{\underline{q},\underline{u}}$, it follows that the Liouville number $\eta=t-\xi$ does not belong to $\mathsf{S}_{\underline{q},\underline{u}}$.

One defines a reflexive and symmetric relation R between two Liouville numbers by $\xi R\eta$ if $\{\xi,\eta\}$ is a Liouville set. The equivalence relation which is induced by R is trivial, as shown by the next result, which is a consequence of Theorem 2.

Corollary 2. *Let ξ and η be Liouville numbers. Then there exists a subset ϑ of $\mathbb{L}$ having the power of continuum such that, for each such $\varrho\in\vartheta$, both sets $\{\xi,\varrho\}$ and $\{\eta,\varrho\}$ are Liouville sets.*

In [3], É. Maillet introduces the definition of Liouville numbers *corresponding* to a given Liouville number. However this definition depends on the choice of a given sequence $\underline{q}$ giving the rational approximations. This is why we start with a sequence $\underline{q}$ instead of starting with a given Liouville number.

The intersection of two nonempty Liouville sets may be empty. More generally, we show that there are uncountably many Liouville sets $\mathsf{S}_{\underline{q}}$ with pairwise empty intersections.

Proposition 1. *For $0 < \tau < 1$, define $\underline{q}^{(\tau)}$ as the sequence $(q_n^{(\tau)})_{n\geq 1}$ with*

$$q_n^{(\tau)} = 2^{n!\lfloor n^\tau \rfloor} \qquad (n \geq 1).$$

Then the sets $\mathsf{S}_{\underline{q}^{(\tau)}}$, $0 < \tau < 1$, are nonempty (hence uncountable) and pairwise disjoint.

To prove that a real number is not a Liouville number is most often difficult. But to prove that a given real number does not belong to some Liouville set S is easier. If $\underline{q}'$ is a subsequence of a sequence $\underline{q}$, one may expect that $\mathsf{S}_{\underline{q}'}$ may often contain strictly $\mathsf{S}_{\underline{q}}$. Here is an example:

Proposition 2. *Define the sequences $\underline{q}$, $\underline{q}'$ and $\underline{q}''$ by*

$$q_n = 2^{n!}, \quad q'_n = q_{2n} = 2^{(2n)!} \quad \textit{and} \quad q''_n = q_{2n+1} = 2^{(2n+1)!} \qquad (n \geq 1),$$

so that $\underline{q}$ is the increasing sequence deduced from the union of $\underline{q}'$ and $\underline{q}''$. Let λ_n be a sequence of positive integers such that

$$\lim_{n\to\infty} \lambda_n = \infty \quad \textit{and} \quad \lim_{n\to\infty} \frac{\lambda_n}{n} = 0.$$

Then the number

$$\xi := \sum_{n\geq 1} \frac{1}{2^{(2n-1)!\lambda_n}}$$

belongs to $\mathsf{S}_{\underline{q}'}$ but not to $\mathsf{S}_{\underline{q}}$. Moreover

$$\mathsf{S}_{\underline{q}} = \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}.$$

When $\underline{q}$ is the increasing sequence deduced from the union of $\underline{q}'$ and $\underline{q}''$, we always have $\mathsf{S}_{\underline{q}} \subset \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$. Proposition 1 gives an example where $\mathsf{S}_{\underline{q}'} \neq \emptyset$ and $\mathsf{S}_{\underline{q}''} \neq \emptyset$, while $\mathsf{S}_{\underline{q}}$ is the empty set. In the example from Proposition 2, the set $\mathsf{S}_{\underline{q}}$ coincides with $\mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$. This is not always the case.

Proposition 3. *There exist two increasing sequences $\underline{q}'$ and $\underline{q}''$ of positive integers with union $\underline{q}$ such that $\mathsf{S}_{\underline{q}}$ is a strict nonempty subset of $\mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$.*

Also, we prove that given any increasing sequence $\underline{q}$, there exists a subsequence $\underline{q}'$ of $\underline{q}$ such that $\mathsf{S}_{\underline{q}}$ is a strict subset of $\mathsf{S}_{\underline{q}'}$. More generally, we prove

Proposition 4. *Let $\underline{u} = (u_n)_{n\geq 1}$ be a sequence of positive real numbers such that for every $n \geq 1$, we have $\sqrt{u_{n+1}} \leq u_n + 1 \leq u_{n+1}$. Then any increasing sequence $\underline{q}$ of positive integers has a subsequence $\underline{q}'$ for which $\mathsf{S}_{\underline{q}',\underline{u}}$ strictly contains $\mathsf{S}_{\underline{q},\underline{u}}$. In particular, any increasing sequence $\underline{q}$ of positive integers has a subsequence $\underline{q}'$ for which $\mathsf{S}_{\underline{q}'}$ strictly contains $\mathsf{S}_{\underline{q}}$.*

Proposition 5. *The sets* $\mathsf{S}_{\underline{q},\underline{u}}$ *are not* G_δ*-subsets of* $\mathbf{R}$. *If they are nonempty, then they are dense in* $\mathbf{R}$.

The proof of Lemma 1 is given in section 2, the proof of Theorem 1 in section 3, the proof of Theorem 2 in section 4, the proof of Corollary 2 in section 5. The proofs of Propositions 1, 2, 3 and 4 are given in section 6 and the proof of Proposition 5 is given in section 7.

2. Proof of lemma 1

Proof of Lemma 1. Given $\underline{q}$ and $\underline{u}$, define inductively a sequence of positive integers $(m_n)_{n\geq 1}$ as follows. Let m_1 be the least integer $m \geq 1$ such that $u_m > 1$. Once $m_1, \dots, m_{n-1}$ are known, define m_n as the least integer $m > m_{n-1}$ for which $u_m > n$. Consider the subsequence $\underline{q}'$ of $\underline{q}$ defined by $q'_n = q_{m_n}$. Then $\mathsf{S}_{\underline{q},\underline{u}} \subset \mathsf{S}_{\underline{q}'}$, hence $\mathsf{S}_{\underline{q},\underline{u}}$ is a Liouville set. □

Remark 1. In the definition of a Liouville set, if assumption (1) is satisfied for some κ_1, then it is also satisfied with κ_1 replaced by any $\kappa'_1 > \kappa_1$. Hence there is no loss of generality to assume $\kappa_1 > 1$. Then, in this definition, one could add to (1) the condition $q_n \leq b_n$. Indeed, if, for some n, we have $b_n < q_n$, then we set

$$b'_n = \left\lceil \frac{q_n}{b_n} \right\rceil b_n,$$

so that

$$q_n \leq b'_n \leq q_n + b_n \leq 2q_n.$$

Denote by a_n the nearest integer to $b_n\xi$ and set

$$a'_n = \left\lceil \frac{q_n}{b_n} \right\rceil a_n.$$

Then, for $\kappa'_2 < \kappa_2$ and, for sufficiently large n, we have

$$\left|b'_n\xi - a'_n\right| = \left\lceil \frac{q_n}{b_n} \right\rceil \left|b_n\xi - a_n\right| \leq \frac{q_n}{q_n^{\kappa_2 n}} \leq \frac{1}{(q_n)^{\kappa'_2 n}}.$$

Hence condition (1) can be replaced by

$$q_n \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 n}}.$$

Also, one deduces from Theorem 2, that the sequence $\big(b_n\big)_{n\geq 1}$ is increasing for sufficiently large n. Note also that in the same way we can assume that

$$q_n \leq b_n \leq q_n^{\kappa_1} \text{ and } \|b_n\xi\| \leq \frac{1}{q_n^{\kappa_2 u_n}}.$$

3. Proof of Theorem 1

We first prove the following:

Lemma 2. *Let* $\underline{q}$ *be an increasing sequence of positive integers and* $\underline{u} = (u_n)_{n\geq 1}$ *be an increasing sequence of real numbers. Let* $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$. *Then* $1/\xi \in \mathsf{S}_{\underline{q},\underline{u}}$.

As a consequence, if S is a Liouville set, then, for any $\xi \in \mathsf{S}$, the set $\mathsf{S} \cup \{1/\xi\}$ is a Liouville set.

Proof of Lemma 2. Let $\underline{q} = (q_n)_{n\ge 1}$ be an increasing sequence of positive integers such that, for sufficiently large n,

$$\| \, b_n \xi \| \le q_n^{-u_n},$$

where $b_n \le q_n^{\kappa_1}$. Write $\|b_n\xi\| = |b_n\xi - a_n|$ with $a_n \in \mathbf{Z}$. Since $\xi \notin \mathbf{Q}$, the sequence $(|a_n|)_{n\ge 1}$ tends to infinity; in particular, for sufficiently large n, we have $a_n \neq 0$. Writing

$$\frac{1}{\xi} - \frac{b_n}{a_n} = \frac{-b_n}{\xi a_n}\left(\xi - \frac{a_n}{b_n}\right),$$

one easily checks that, for sufficiently large n,

$$\| \, |a_n|\xi^{-1}\| \le |a_n|^{-u_n/2} \quad \text{and} \quad 1 \le |a_n| < b_n^2 \le q_n^{2\kappa_1}.$$

$\square$

Proof of Theorem 1. Let us check that for ξ and ξ' in $\mathbf{Q} \cup \mathsf{S}_{\underline{q},\underline{u}}$, we have $\xi - \xi' \in \mathbf{Q} \cup \mathsf{S}_{\underline{q},\underline{u}}$ and $\xi\xi' \in \mathbf{Q} \cup \mathsf{S}_{\underline{q},\underline{u}}$. Clearly, it suffices to check
(1) For ξ in $\mathsf{S}_{\underline{q},\underline{u}}$ and ξ' in $\mathbf{Q}$, we have $\xi - \xi' \in \mathsf{S}_{\underline{q},\underline{u}}$ and $\xi\xi' \in \mathsf{S}_{\underline{q},\underline{u}}$.
(2) For ξ in $\mathsf{S}_{\underline{q},\underline{u}}$ and ξ' in $\mathsf{S}_{\underline{q},\underline{u}}$ with $\xi - \xi' \notin \mathbf{Q}$, we have $\xi - \xi' \in \mathsf{S}_{\underline{q},\underline{u}}$.
(3) For ξ in $\mathsf{S}_{\underline{q},\underline{u}}$ and ξ' in $\mathsf{S}_{\underline{q},\underline{u}}$ with $\xi\xi' \notin \mathbf{Q}$, we have $\xi\xi' \in \mathsf{S}_{\underline{q},\underline{u}}$.

The idea of the proof is as follows. When $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$ is approximated by a_n/b_n and when $\xi' = r/s \in \mathbf{Q}$, then $\xi - \xi'$ is approximated by $(sa_n - rb_n)/b_n$ and $\xi\xi'$ by ra_n/sb_n. When $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$ is approximated by a_n/b_n and $\xi' \in \mathsf{S}_{\underline{q},\underline{u}}$ by a'_n/b'_n, then $\xi - \xi'$ is approximated by $(a_nb'_n - a'_nb_n)/b_nb'_n$ and $\xi\xi'$ by $a_na'_n/b_nb'_n$. The proofs which follow amount to writing down carefully these simple observations.

Let $\xi'' = \xi - \xi'$ and $\xi^* = \xi\xi'$. Then the sequence (a''_n) and (b''_n) are corresponding to ξ''. Similarly (a^*_n) and (b^*_n) correspond to ξ^*.

Here is the proof of (1). Let $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$ and $\xi' = r/s \in \mathbf{Q}$, with r and s in $\mathbf{Z}$, $s > 0$. There are two constants κ_1 and κ_2 and there are sequences of rational integers $\big(a_n\big)_{n\ge 1}$ and $\big(b_n\big)_{n\ge 1}$ such that

$$1 \le b_n \le q_n^{\kappa_1} \quad \text{and} \quad 0 < \big|b_n\xi - a_n\big| \le \frac{1}{q_n^{\kappa_2 u_n}}\cdot$$

Let $\tilde\kappa_1 > \kappa_1$ and $\tilde\kappa_2 < \kappa_2$. Then,

$$\begin{aligned} b''_n &= b^*_n = sb_n,\\ a''_n &= sa_n - rb_n,\\ a^*_n &= ra_n. \end{aligned}$$

Then one easily checks that, for sufficiently large n, we have

$$0 < \big|b''_n\xi'' - a''_n\big| = s\,\big|b_n\xi - a_n\big| \le \frac{1}{q_n^{\kappa''_2 u_n}},$$

$$0 < \big|b^*_n\xi^* - a^*_n\big| = |r|\,\big|b_n\xi - a_n\big| \le \frac{1}{q_n^{\kappa^*_2 u_n}}\cdot$$

Here is the proof of (2) and (3). Let ξ and ξ' be in $\mathsf{S}_{\underline{q},\underline{u}}$. There are constants κ_1', κ_2', κ_1'' and κ_2'' and there are sequences of rational integers $\bigl(a_n\bigr)_{n\geq 1}$, $\bigl(b_n\bigr)_{n\geq 1}$, $\bigl(a_n'\bigr)_{n\geq 1}$ and $\bigl(b_n'\bigr)_{n\geq 1}$ such that

$$1 \leq b_n \leq q_n^{\kappa_1'} \quad \text{and} \quad 0 < \bigl|b_n\xi - a_n\bigr| \leq \frac{1}{q_n^{\kappa_2' u_n}},$$

$$1 \leq b_n' \leq q_n^{\kappa_1''} \quad \text{and} \quad 0 < \bigl|b_n'\xi' - a_n'\bigr| \leq \frac{1}{q_n^{\kappa_2'' u_n}}\cdot$$

Define $\tilde{\kappa}_1 = \kappa_1' + \kappa_1''$ and let $\tilde{\kappa}_2 > 0$ satisfy $\tilde{\kappa}_2 < \min\{\kappa_2', \kappa_2''\}$. Set

$$\begin{aligned} b_n'' &= b_n^* = b_n b_n', \\ a_n'' &= a_n b_n' - b_n a_n', \\ a_n^* &= a_n a_n'. \end{aligned}$$

Then for sufficiently large n, we have

$$b_n''\xi'' - a_n'' = b_n'\bigl(b_n\xi - a_n\bigr) - b_n\bigl(b_n'\xi' - a_n'\bigr)$$

and

$$b_n^*\xi^* - a_n^* = b_n\xi\left(b_n'\xi' - a_n'\right) + a_n'\left(b_n\xi - a_n\right),$$

hence

$$\bigl|b_n''\xi'' - a_n''\bigr| \leq \frac{1}{q_n^{\tilde{\kappa}_2 u_n}}$$

and

$$\bigl|b_n^*\xi^* - a_n^*\bigr| \leq \frac{1}{q_n^{\tilde{\kappa}_2 u_n}}\cdot$$

Also we have

$$1 \leq b_n'' \leq q_n^{\tilde{\kappa}_1} \quad \text{and} \quad 1 \leq b_n^* \leq q_n^{\tilde{\kappa}_1}.$$

The assumption $\xi - \xi' \notin \mathbf{Q}$ (respectively $\xi\xi' \notin \mathbf{Q}$) implies $b_n''\xi'' \neq a_n''$ (respectively, $b_n^*\xi^* \neq a_n^*$). Hence $\xi - \xi'$ and $\xi\xi'$ are in $\mathsf{S}_{\underline{q},\underline{u}}$. This completes the proof of (2) and (3).

It follows from (1), (2) and (3) that $\mathbf{Q} \cup \mathsf{S}_{\underline{q},\underline{u}}$ is a ring.

Finally, if $\xi \in \mathbf{Q}\cup\mathsf{S}_{\underline{q},\underline{u}}$ is not 0, then $1/\xi \in \mathbf{Q}\cup\mathsf{S}_{\underline{q},\underline{u}}$, by Lemma 2. This completes the proof of Theorem 1. □

Remark 2. Since the field $\mathsf{K}_{\underline{q},\underline{u}}$ does not contain irrational algebraic numbers, 2 is not a square in $\mathsf{K}_{\underline{q},\underline{u}}$. For $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$, it follows that $\eta = 2\xi^2$ is an element in $\mathsf{S}_{\underline{q},\underline{u}}$ which is not the square of an element in $\mathsf{S}_{\underline{q},\underline{u}}$. According to [1], we can write $\sqrt{2} = \xi_1\xi_2$ with two Liouville numbers ξ_1, ξ_2; then the set $\{\xi_1, \xi_2\}$ is not a Liouville set.

Let N be a positive integer such that N cannot be written as a sum of two squares of an integer. Let us show that, for $\varrho \in \mathsf{S}_{\underline{q},\underline{u}}$, the Liouville number $N\varrho^2 \in \mathsf{S}_{\underline{q},\underline{u}}$ is not the sum of two squares of elements in $\mathsf{S}_{\underline{q},\underline{u}}$. Dividing by ϱ^2, we are reduced to show that the equation $N = \xi^2 + (\xi')^2$ has no solution (ξ, ξ') in $\mathsf{S}_{\underline{q},\underline{u}} \times \mathsf{S}_{\underline{q},\underline{u}}$. Otherwise, we would have, for suitable positive constants κ_1 and κ_2,

$$\left|\xi - \frac{a_n}{b_n}\right| \leq \frac{1}{q_n^{\kappa_2 u_n + 1}}, \qquad 1 \leq b_n \leq q_n^{\kappa_1},$$

$$\left|\xi' - \frac{a_n'}{b_n'}\right| \leq \frac{1}{q_n^{\kappa_2 u_n + 1}}, \qquad 1 \leq b_n' \leq q_n^{\kappa_1},$$

hence

$$\left|\xi^2 - \frac{a_n^2}{b_n^2}\right| \le \frac{2|\xi|+1}{q_n^{\kappa_2 u_n+1}}, \qquad \left|(\xi')^2 - \frac{(a_n')^2}{(b_n')^2}\right| \le \frac{2|\xi'|+1}{q_n^{\kappa_2 u_n+1}}$$

and

$$\left|\xi^2 + (\xi')^2 - \frac{\big(a_n b_n'\big)^2 + \big(a_n' b_n\big)^2}{\big(b_n b_n'\big)^2}\right| \le \frac{2(|\xi|+|\xi'|+1)}{q_n^{\kappa_2 u_n+1}}.$$

Using $\xi^2 + (\xi')^2 = N$, we deduce

$$\left|N\big(b_n b_n'\big)^2 - \big(a_n b_n'\big)^2 - \big(a_n' b_n\big)^2\right| < 1.$$

The left hand side is an integer, hence it is 0:

$$N\big(b_n b_n'\big)^2 = \big(a_n b_n'\big)^2 + \big(a_n' b_n\big)^2.$$

This is impossible, since the equation $x^2 + y^2 = Nz^2$ has no solution in positive rational integers.

Therefore, if we write $N = \xi^2 + (\xi')^2$ with two Liouville numbers ξ, ξ', which is possible by the above mentioned result from P. Erdős [1], then the set $\{\xi, \xi'\}$ is not a Liouville set.

4. Proof of Theorem 2

We first prove the following lemma which will be required for the proof of part (ii) of Theorem 2.

Lemma 3. *Let ξ be a real number, n, q and q' be positive integers with $n \ge 2$. Assume that there exist rational integers p and p' such that $p/q \ne p'/q'$ and*

$$|q\xi - p| \le \frac{1}{q^{u_n}}, \quad |q'\xi - p'| \le \frac{1}{(q')^{u_n+1}}.$$

Then we have

$$\text{either} \quad q' + 1 > q^{u_n} \quad \text{or} \quad q \ge (q')^{u_n}.$$

Proof of Lemma 3. From the assumptions we deduce

$$\frac{1}{qq'} \le \frac{|pq' - p'q|}{qq'} \le \left|\xi - \frac{p}{q}\right| + \left|\xi - \frac{p'}{q'}\right| \le \frac{1}{q^{u_n+1}} + \frac{1}{(q')^{u_n+2}},$$

hence

$$q^{u_n}(q')^{u_n+1} \le (q')^{u_n+2} + q^{u_n+1}.$$

If $q < q'$, we deduce

$$q^{u_n} \le q' + \left(\frac{q}{q'}\right)^{u_n+1} < q' + 1.$$

Assume now $q \ge q'$. Since the conclusion of Lemma 3 is trivial if $q' = 1$, we assume $q' \ge 2$. Since $n \ge 2$, we have $u_n \ge 2$. From

$$q^{u_n}(q')^{u_n+1} \le (q')^{u_n+2} + q^{u_n+1} \le (q')^2 q^{u_n} + q^{u_n+1}$$

we deduce

$$(q')^{u_n+1} - (q')^2 \le q.$$

From

$$(q')^3 - (q')^2 \le q$$

we deduce

$$(q')^2 \le \frac{q}{q'-1} \le q(q'-1).$$

which implies

$$\frac{q}{q'} + q' \le q$$

Finally,

$$(q')^{u_n} \le \frac{q}{q'} + q' \le q.$$

□

Proof of Theorem 2. Suppose $\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} = 0$. Then, we get

$$\lim_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} = 0.$$

Suppose $\mathsf{S}_{\underline{q},\underline{u}} \neq \emptyset$. Let $\xi \in \mathsf{S}_{\underline{q},\underline{u}}$. From Remark 1, it follows that there exists a sequence $\big(b_n\big)_{n\ge1}$ of positive integers and there exist two positive constants κ_1 and κ_2 such that, for any sufficiently large n,

$$q_n \le b_n \le q_n^{\kappa_1} \text{ and } \|b_n\xi\| \le q_n^{-\kappa_2 u_n}.$$

Let n_0 be an integer $\ge \kappa_1$ such that these inequalities are valid for $n \ge n_0$ and such that, for $n \ge n_0$, $q_{n+1}^{\kappa_1} < q_n^{u_n}$ (by the assumption). Since the sequence $(q_n)_{n\ge1}$ is increasing, we have $q_n^{\kappa_1} < q_{n+1}^{u_n}$ for $n \ge n_0$. From the choice of n_0 we deduce

$$b_{n+1} \le q_{n+1}^{\kappa_1} < q_n^{u_n} \le b_n^{u_n}$$

and

$$b_n \le q_n^{\kappa_1} < q_{n+1}^{u_n} \le b_{n+1}^{u_n}$$

for any $n \ge n_0$. Denote by a_n (respectively a_{n+1}) the nearest integer to ξb_n (respectively to ξb_{n+1}). Lemma 3 with q replaced by b_n and q' by b_{n+1} implies that for each $n \ge n_0$,

$$\frac{a_n}{b_n} = \frac{a_{n+1}}{b_{n+1}}\cdot$$

This contradicts the assumption that ξ is irrational. This proves that $\mathsf{S}_{\underline{q},\underline{u}} = \emptyset$. Conversely, assume

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{u_n \log q_n} > 0.$$

Then there exists $\vartheta > 0$ and there exists a sequence $(N_\ell)_{\ell\ge1}$ of positive integers such that

$$q_{N_\ell} > q_{N_\ell - 1}^{\vartheta u_{N_\ell - 1}}$$

for all $\ell \ge 1$. Define a sequence $(c_\ell)_{\ell\ge1}$ of positive integers by

$$2^{c_\ell} \le q_{N_\ell} < 2^{c_\ell+1}.$$

Let $\underline{e} = (e_\ell)_{\ell\ge1}$ be a sequence of elements in $\{-1, 1\}$. Define

$$\xi_{\underline{e}} = \sum_{\ell\ge1} \frac{e_\ell}{2^{c_\ell}}\cdot$$

It remains to check that $\xi_{\underline{e}} \in \mathsf{S}_{\underline{q},\underline{u}}$ and that distinct $\underline{e}$ produce distinct $\xi_{\underline{e}}$.

Let $\kappa_1 = 1$ and let κ_2 be in the interval $0 < \kappa_2 < \vartheta$. For sufficiently large n, let ℓ be the integer such that $N_{\ell-1} \le n < N_\ell$. Set

$$b_n = 2^{c_{\ell-1}}, \quad a_n = \sum_{h=1}^{\ell-1} e_h 2^{c_{\ell-1}-c_h}, \quad r_n = \frac{a_n}{b_n}\cdot$$

We have

$$\frac{1}{2^{c_\ell}} < \left|\xi_{\underline{e}} - r_n\right| = \left|\xi_{\underline{e}} - \sum_{h\geq \ell} \frac{e_h}{2^{c_h}}\right| \leq \frac{2}{2^{c_\ell}}\cdot$$

Since $\kappa_2 < \vartheta$, n is sufficiently large and $n \leq N_\ell - 1$, we have

$$4q_n^{\kappa_2 u_n} \leq 4q_{N_\ell -1}^{\kappa_2 u_{N_\ell -1}} \leq q_{N_\ell},$$

hence

$$\frac{2}{2^{c_\ell}} < \frac{4}{q_{N_\ell}} < \frac{1}{q_n^{\kappa_2 u_n}}$$

for sufficiently large n. This proves $\xi_{\underline{e}} \in \mathsf{S}_{\underline{q},\underline{u}}$ and hence $\mathsf{S}_{\underline{q},\underline{u}}$ is not empty.

Finally, if $\underline{e}$ and $\underline{e}'$ are two elements of $\{-1,+1\}^{\mathbf{N}}$ for which $e_h = e'_h$ for $1 \leq h < \ell$ and, say, $e_\ell = -1$, $e'_\ell = 1$, then

$$\xi_{\underline{e}} < \sum_{h=1}^{\ell-1} \frac{e_h}{2^{c_h}} < \xi_{\underline{e}'},$$

hence $\xi_{\underline{e}} \neq \xi_{\underline{e}'}$. This completes the proof of Theorem 2. □

5. Proof of Corollary 2

The proof of Corollary 2 as a consequence of Theorem 2 relies on the following elementary lemma.

Lemma 4. *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two increasing sequences of positive integers. Then there exists an increasing sequence of positive integers $(q_n)_{n\geq 1}$ satisfying the following properties:*
(i) The sequence $(q_{2n})_{n\geq 1}$ is a subsequence of the sequence $(a_n)_{n\geq 1}$.
(ii) The sequence $(q_{2n+1})_{n\geq 0}$ is a subsequence of the sequence $(b_n)_{n\geq 1}$.
(iii) For $n \geq 1$, $q_{n+1} \geq q_n^n$.

Proof of Lemma 4. We construct the sequence $(q_n)_{n\geq 1}$ inductively, starting with $q_1 = b_1$ and with q_2 the least integer a_i satisfying $a_i \geq b_1$. Once q_n is known for some $n \geq 2$, we take for q_{n+1} the least integer satisfying the following properties:
- $q_{n+1} \in \{a_1, a_2, \dots\}$ if n is odd, $q_{n+1} \in \{b_1, b_2, \dots\}$ if n is even.
- $q_{n+1} \geq q_n^n$. □

Proof of Corollary 2. Let ξ and η be Liouville numbers. There exist two sequences of positive integers $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$, which we may assume to be increasing, such that

$$\|a_n\xi\| \leq a_n^{-n} \quad \text{and} \quad \|b_n\eta\| \leq b_n^{-n}$$

for sufficiently large n. Let $\underline{q} = (q_n)_{n\geq 1}$ be an increasing sequence of positive integers satisfying the conclusion of Lemma 4. According to Theorem 2, the Liouville set $\mathsf{S}_{\underline{q}}$ is not empty. Let $\varrho \in \mathsf{S}_{\underline{q}}$. Denote by $\underline{q}'$ the subsequence $(q_2, q_4, \dots, q_{2n}, \dots)$ of $\underline{q}$ and by $\underline{q}''$ the subsequence $(q_1, q_3, \dots, q_{2n+1}, \dots)$. We have $\varrho \in \mathsf{S}_{\underline{q}} = \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$. Since the sequence $(a_n)_{n\geq 1}$ is increasing, we have $q_{2n} \geq a_n$, hence $\xi \in \mathsf{S}_{\underline{q}'}$. Also, since the sequence $(b_n)_{n\geq 1}$ is increasing, we have $q_{2n+1} \geq b_n$, hence $\eta \in \mathsf{S}_{\underline{q}''}$. Finally, ξ and ϱ belong to the Liouville set $\mathsf{S}_{\underline{q}'}$, while η and ϱ belong to the Liouville set $\mathsf{S}_{\underline{q}''}$. □

6. Proofs of Propositions 1, 2, 3 and 4

Proof of Proposition 1. The fact that for $0 < \tau < 1$ the set $\mathsf{S}_{\underline{q}^{(\tau)}}$ is not empty follows from Theorem 2, since

$$\lim_{n\to\infty} \frac{\log q_{n+1}^{(\tau)}}{n \log q_n^{(\tau)}} = 1.$$

In fact, if $(e_n)_{n\geq 1}$ is a bounded sequence of integers with infinitely many nonzero terms, then

$$\sum_{n\geq 1} \frac{e_n}{q_n^{(\tau)}} \in \mathsf{S}_{\underline{q}^{(\tau)}}.$$

Let $0 < \tau_1 < \tau_2 < 1$. For $n \geq 1$, define

$$q_{2n} = q_n^{(\tau_1)} = 2^{n!\lfloor n^{\tau_1}\rfloor} \quad \text{and} \quad q_{2n+1} = q_n^{(\tau_2)} = 2^{n!\lfloor n^{\tau_2}\rfloor}.$$

One easily checks that $(q_m)_{m\geq 1}$ is an increasing sequence with

$$\frac{\log q_{2n+1}}{n \log q_{2n}} \to 0 \quad \text{and} \quad \frac{\log q_{2n+2}}{n \log q_{2n+1}} \to 0.$$

From Theorem 2 one deduces $\mathsf{S}_{\underline{q}^{(\tau_1)}} \cap \mathsf{S}_{\underline{q}^{(\tau_2)}} = \emptyset$. □

Proof of Proposition 2. For sufficiently large n, define

$$a_n = \sum_{m=1}^{n} 2^{(2n)!-(2m-1)!\lambda_m}.$$

Then

$$\frac{1}{q_{2n}^{(2n+1)\lambda_{n+1}}} < \xi - \frac{a_n}{q_{2n}} = \sum_{m\geq n+1} \frac{1}{2^{(2m-1)!\lambda_m}} \leq \frac{2}{q_{2n}^{(2n+1)\lambda_{n+1}}}\cdot$$

The right inequality with the lower bound $\lambda_{n+1} \geq 1$ proves that $\xi \in \mathsf{S}_{\underline{q}'}$.

Let κ_1 and κ_2 be positive numbers, n a sufficiently large integer, s an integer in the interval $q_{2n+1} \leq s \leq q_{2n+1}^{\kappa_1}$ and r an integer. Since $\lambda_{n+1} < \kappa_2 n$ for sufficiently large n, we have

$$q_{2n}^{(2n+1)\lambda_{n+1}} < q_{2n}^{\kappa_2 n(2n+1)} = q_{2n+1}^{\kappa_2 n} \leq s^{\kappa_2 n}.$$

Therefore, if $r/s = a_n/q_{2n}$, then

$$\left|\xi - \frac{r}{s}\right| = \left|\xi - \frac{a_n}{q_{2n}}\right| > \frac{1}{q_{2n}^{(2n+1)\lambda_{n+1}}} > \frac{1}{s^{\kappa_2 n}}\cdot$$

On the other hand, for $r/s \neq a_n/q_{2n}$, we have

$$\left|\xi - \frac{r}{s}\right| \geq \left|\frac{a_n}{q_{2n}} - \frac{r}{s}\right| - \left|\xi - \frac{a_n}{q_{2n}}\right| \geq \frac{1}{q_{2n}s} - \frac{2}{q_{2n}^{(2n+1)\lambda_{n+1}}}\cdot$$

Since $\lambda_n \to \infty$, for sufficiently large n we have

$$4q_{2n}s \leq 4q_{2n}q_{2n+1}^{\kappa_1} = 4q_{2n}^{1+\kappa_1(2n+1)} \leq q_{2n}^{(2n+1)\lambda_{n+1}}$$

hence

$$\frac{2}{q_{2n}^{(2n+1)\lambda_{n+1}}} \leq \frac{1}{2q_{2n}s}\cdot$$

Further

$$2q_{2n} < q_{2n+1} < q_{2n+1}^{\kappa_2 n-1} \leq s^{\kappa_2 n-1}.$$

Therefore

$$\left|\xi - \frac{r}{s}\right| \geq \frac{1}{2q_{2n}s} > \frac{1}{s^{\kappa_2 n}},$$

which shows that $\xi \notin \mathsf{S}_{\underline{q}''}$. □

Proof of Proposition 3. Let $(\lambda_s)_{s\geq 0}$ be a strictly increasing sequence of positive rational integers with $\lambda_0 = 1$. Define two sequences $(n'_k)_{k\geq 1}$ and $(n''_h)_{h\geq 1}$ of positive integers as follows. The sequence $(n'_k)_{k\geq 1}$ is the increasing sequence of the positive integers n for which there exists $s \geq 0$ with $\lambda_{2s} \leq n < \lambda_{2s+1}$, while $(n''_h)_{h\geq 1}$ is the increasing sequence of the positive integers n for which there exists $s \geq 0$ with $\lambda_{2s+1} \leq n < \lambda_{2s+2}$.

For $s \geq 0$ and $\lambda_{2s} \leq n < \lambda_{2s+1}$, set

$$k = n - \lambda_{2s} + \lambda_{2s-1} - \lambda_{2s-2} + \cdots + \lambda_1.$$

Then $n = n'_k$.

For $s \geq 0$ and $\lambda_{2s+1} \leq n < \lambda_{2s+2}$, set

$$h = n - \lambda_{2s+1} + \lambda_{2s} - \lambda_{2s-1} + \cdots - \lambda_1 + 1.$$

Then $n = n''_h$.

For instance, when $\lambda_s = s+1$, the sequence $(n'_k)_{k\geq 1}$ is the sequence $(1, 3, 5\dots)$ of odd positive integers, while $(n''_h)_{h\geq 1}$ is the sequence $(2, 4, 6\dots)$ of even positive integers. Another example is $\lambda_s = s!$, which occurs in the paper [1] by P. Erdős.

In general, for $n = \lambda_{2s}$, we write $n = n'_{k(s)}$ where

$$k(s) = \lambda_{2s-1} - \lambda_{2s-2} + \cdots + \lambda_1 < \lambda_{2s-1}.$$

Notice that $\lambda_{2s} - 1 = n''_h$ with $h = \lambda_{2s} - k(s)$.

Next, define two increasing sequences $(d_n)_{n\geq 1}$ and $\underline{q} = (q_n)_{n\geq 1}$ of positive integers by induction, with $d_1 = 2$,

$$d_{n+1} = \begin{cases} kd_n & \text{if } n = n'_k, \\ hd_n & \text{if } n = n''_h \end{cases}$$

for $n \geq 1$ and $q_n = 2^{d_n}$. Finally, let $\underline{q}' = (q'_k)_{k\geq 1}$ and $\underline{q}'' = (q''_h)_{h\geq 1}$ be the two subsequences of $\underline{q}$ defined by

$$q'_k = q_{n'_k}, \quad k \geq 1, \qquad q''_h = q_{n''_h}, \quad h \geq 1.$$

Hence $\underline{q}$ is the union of these two subsequences. Now we check that the number

$$\xi = \sum_{n\geq 1} \frac{1}{q_n}$$

belongs to $\mathsf{S}_{\underline{q}'} \bigcap \mathsf{S}_{\underline{q}''}$. Note that by Theorem 2 that $\mathsf{S}_{\underline{q}} \neq \emptyset$ as $\mathsf{S}_{\underline{q}'} \neq \emptyset$ and $\mathsf{S}_{\underline{q}''} \neq \emptyset$. Define

$$a_n = \sum_{m=1}^{n} 2^{d_n - d_m}.$$

Then

$$\frac{1}{q_{n+1}} < \xi - \frac{a_n}{q_n} = \sum_{m\geq n+1} \frac{1}{q_m} < \frac{2}{q_{n+1}}.$$

If $n = n'_k$, then

$$\left|\xi - \frac{a_{n'_k}}{q'_k}\right| < \frac{2}{(q'_k)^k},$$

while if $n = n''_h$, then

$$\left|\xi - \frac{a_{n''_h}}{q''_h}\right| < \frac{2}{(q''_h)^h}.$$

This proves that $\xi \in \mathsf{S}_{\underline{q}'} \cap \mathsf{S}_{\underline{q}''}$.

Now, we choose $\lambda_s = 2^{2^s}$ for $s \geq 2$ and we prove that ξ does not belong to $\mathsf{S}_{\underline{q}}$. Notice that $\lambda_{2s-1} = \sqrt{\lambda_{2s}}$. Let $n = \lambda_{2s} = n'_{k(s)}$. We have $k(s) < \sqrt{\lambda_{2s}}$ and

$$\left|\xi - \frac{a_n}{q_n}\right| > \frac{1}{q_{n+1}} = \frac{1}{q_n^{k(s)}} > \frac{1}{q_n^{\sqrt{n}}}.$$

Let κ_1 and κ_2 be two positive real numbers and assume s is sufficiently large. Further, let $u/v \in \mathbf{Q}$ with $v \leq q_n^{\kappa_1}$. If $u/v = a_n/q_n$, then

$$\left|\xi - \frac{u}{v}\right| = \left|\xi - \frac{a_n}{q_n}\right| > \frac{1}{q_n^{\sqrt{n}}} > \frac{1}{q_n^{\kappa_2 n}}.$$

On the other hand, if $u/v \neq a_n/q_n$, then

$$\left|\xi - \frac{u}{v}\right| \geq \left|\frac{u}{v} - \frac{a_n}{q_n}\right| - \left|\xi - \frac{a_n}{q_n}\right|$$

with

$$\left|\frac{u}{v} - \frac{a_n}{q_n}\right| \geq \frac{1}{vq_n} \geq \frac{1}{q_n^{\kappa_1+1}} > \frac{2}{q_n^{\sqrt{n}}}$$

and

$$\left|\xi - \frac{a_n}{q_n}\right| < \frac{1}{q_n^{\sqrt{n}}}.$$

Hence,

$$\left|\xi - \frac{u}{v}\right| > \frac{1}{q_n^{\sqrt{n}}} > \frac{1}{q_n^{\kappa_2 n}}.$$

This proves Proposition 3. □

Proof of Proposition 4. Let $\underline{u} = (u_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\sqrt{u_{n+1}} \leq u_n + 1 \leq u_{n+1}$. We prove more precisely that for any sequence $\underline{q}$ such that $q_{n+1} > q_n^{u_n}$ for all $n \geq 1$, the sequence $\underline{q}' = (q_{2m+1})_{m\geq 1}$ has $\mathsf{S}_{\underline{q}',\underline{u}} \neq \mathsf{S}_{\underline{q},\underline{u}}$. This implies the proposition, since any increasing sequence has a subsequence satisfying $q_{n+1} > q_n^{u_n}$.

Assuming $q_{n+1} > q_n^{u_n}$ for all $n \geq 1$, we define

$$d_n = \begin{cases} q_n & \text{for even } n, \\ q_{n-1}^{\lfloor\sqrt{u_n}\rfloor} & \text{for odd } n. \end{cases}$$

We check that the number

$$\xi = \sum_{n\geq 1} \frac{1}{d_n}$$

satisfies $\xi \in \mathsf{S}_{\underline{q}',\underline{u}}$ and $\xi \notin \mathsf{S}_{\underline{q},\underline{u}}$.

Set $b_n = d_1 d_2 \cdots d_n$ and

$$a_n = \sum_{m=1}^{n} \frac{b_n}{d_m} = \sum_{m=1}^{n} \prod_{1\le i\le n, i\ne m} d_i,$$

so that

$$\xi - \frac{a_n}{b_n} = \sum_{m\ge n+1} \frac{1}{d_m}\cdot$$

It is easy to check from the definition of d_n and q_n that we have, for sufficiently large n,

$$b_n \le q_1 \cdots q_n \le q_{n-1}^{u_{n-1}} q_n \le q_n^2$$

and

$$\frac{1}{d_{n+1}} \le \xi - \frac{a_n}{b_n} \le \frac{2}{d_{n+1}}\cdot$$

For odd n, since $d_{n+1} = q_{n+1} \ge q_n^{u_n}$, we deduce

$$\left|\xi - \frac{a_n}{b_n}\right| \le \frac{2}{q_n^{u_n}},$$

hence $\xi \in \mathsf{S}_{\underline{q}',\underline{u}}$.

For even n, we plainly have

$$\left|\xi - \frac{a_n}{b_n}\right| > \frac{1}{d_{n+1}} = \frac{1}{q_n^{\lfloor\sqrt{u_{n+1}}\rfloor}}\cdot$$

Let κ_1 and κ_2 be two positive real numbers, and let n be sufficiently large. Let s be a positive integer with $s \le q_n^{\kappa_1}$ and let r be an integer. If $r/s = a_n/b_n$, then

$$\left|\xi - \frac{r}{s}\right| = \left|\xi - \frac{a_n}{b_n}\right| > \frac{1}{q_n^{\kappa_2 u_n}}\cdot$$

Assume now $r/s \ne a_n/b_n$. From

$$\left|\xi - \frac{a_n}{b_n}\right| \le \frac{2}{q_n^{\lfloor\sqrt{u_{n+1}}\rfloor}} \le \frac{1}{2q_n^{\kappa_1+2}},$$

we deduce

$$\frac{1}{q_n^{\kappa_1+2}} \le \frac{1}{sb_n} \le \left|\frac{r}{s} - \frac{a_n}{b_n}\right| \le \left|\xi - \frac{r}{s}\right| + \left|\xi - \frac{a_n}{b_n}\right| \le \left|\xi - \frac{r}{s}\right| + \frac{1}{2q_n^{\kappa_1+2}},$$

hence

$$\left|\xi - \frac{r}{s}\right| \ge \frac{1}{2q_n^{\kappa_1+2}} > \frac{1}{q_n^{\kappa_2 u_n}}\cdot$$

This completes the proof that $\xi \notin \mathsf{S}_{\underline{q},\underline{u}}$. □

7. Proof of Proposition 5

Proof of Proposition 5. If $\mathsf{S}_{\underline{q},\underline{u}}$ is nonempty, let $\gamma \in \mathsf{S}_{\underline{q},\underline{u}}$. By Theorem 1, $\gamma + \mathbf{Q}$ is contained in $\mathsf{S}_{\underline{q},\underline{u}}$, hence $\mathsf{S}_{\underline{q},\underline{u}}$ is dense in $\mathbf{R}$.

Let t be an irrational real number which is not Liouville. Hence $t \notin \mathsf{K}_{\underline{q},\underline{u}}$, and therefore, by Theorem 1, $\mathsf{S}_{\underline{q},\underline{u}} \cap (t + \mathsf{S}_{\underline{q},\underline{u}}) = \emptyset$. This implies that $\mathsf{S}_{\underline{q},\underline{u}}$ is not a G_δ dense subset of $\mathbf{R}$. □

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