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Unique representation of integers with base A

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Abstract. For a given $A \subseteq \mathbb{N}$, we introduce the concept of representing every positive integer uniquely with base A. We also study the order of magnitude of the function $R_A(n)$, where $R_A(n)$ is the number of digits that are needed to represent n with base A.

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1. Introduction. Let $A \subset \mathbb{N}$ be an infinite subset of the set of positive integers and m be the least element of A. Put $S_A = \{1, 2, \ldots, m-1\}$. For any positive integer n_0, let $a(n_0) \in A$ be the largest integer $\leq n_0$. Put $n_1 = n_0 - a(n_0)$ and $a(n_1) \in A$ be the largest integer $\leq n_1$. Then by letting $n_2 = n_1 - a(n_1)$, we proceed as above until we arrive at $n_r = n_{r-1} - a(n_{r-1})$ with $a(n_i) \in A$ for all $i = 0, 1, 2, \ldots, r-1$ and either $n_r \in A$ or $n_r \in S_A$. Thus, we get

$$n_0 = a(n_0) + a(n_1) + \cdots + a(n_r), \tag{1}$$

where $a(n_i) \in A$ for all $i = 0, 1, 2, \ldots, r-1$ and $a(n_r)$ belongs to either A or S_A. Clearly, the above procedure implies that this representation is unique with respect to the base A. Since r depends on n_0 and A, we write $r = R_A(n_0)$. Clearly, $R_A(n_0)$ is the number of digits of n_0 to the base A.

Example 1. (1) Let p be any prime number and A be the set of all positive powers of A, i.e.

$$A = \{p^k \mid k \in \mathbb{N} \cup \{0\}\}.$$

Then in our notation $S_A = \emptyset$ and hence the representation of integers as in (1) is the usual representation of integers with base p. It is easy to see that $R_A(n) \ll \log n$ for all n and $R_A(n) \gg \log n$ for infinitely many n, where the implied constants in both the cases may depend on the prime p.

(2) Let A be the set of all prime numbers. Then by the definition $S_A = \{1\}$. This particular case was first considered by Pillai [8], who proved that $R_A(n) = o(\log n)$. In the same paper, he proved that $R_A(n) \ll \log\log n$, under the validity of the Riemann hypothesis. Later Luca and Thangadurai [7] proved that $R_A(n) \ll \log\log n$ by using the celebrated theorem of Hohesiel [6] rather than the Riemann hypothesis.

Now it is natural to ask, for a given subset A of $\mathbb{N}$, what is the order of $R_A(n)$ for all n sufficiently large?

The following lemma discards some uninteresting subsets A of $\mathbb{N}$.

Lemma 1. *Let $A = \{a_k \mid k \geq 1\} \subseteq \mathbb{N}$. Then*

$$\limsup_{k\to\infty}(a_k - a_{k-1}) = \infty$$

as $k \to \infty$ if and only if

$$\limsup_{n\to\infty} R_A(n) = \infty.$$

Thus, the problem we consider is trivial in the following cases:

(1) Let A be a subset of positive integers with bounded gaps. Then we have

$$R_A(n) \leq C$$

for some positive constant C.

(2) Let A be the set of all integers which are non-negative integral powers of a fixed positive integer m. Then we can see that

$$R_A(n) \ll_m \log n$$

for all n and

$$R_A(n) \gg_m \log n$$

for infinitely many n.

Remark 1. (1) In view of the Lemma 1, we can always assume that $a_k - a_{k-1} \to \infty$ as $k \to \infty$. That is, we consider those subsets A of $\mathbb{N}$ such that the elements of A have arbitrarily long gaps. For instance, if A is chosen to be the set of all primes, then we get $\limsup R_A(n) = \infty$.

(2) The constant C appearing in this paper may not be the same everywhere.

Definition 1. For a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, we define the composition of f to itself n-times, f^n, to be $f\left(f^{n-1}\right)$ with $f^1(x) = f(x)$.

Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be any function such that $f(x) < x$ for all $x \in \mathbb{R}_{\geq 0}$ and $x_0 \in \mathbb{R}_{\geq 0}$ be a given real number. We define a function $t_{x_0} : \mathbb{N} \to \mathbb{R}_{\geq 0}$ as

$$t_{x_0}(n) = \begin{cases} n & \textit{if } n \leq x_0 \\ \ell & \textit{if } n > x_0 \quad \textit{and} \quad f^{\ell}(n) < x_0 < f^{\ell-1}(n). \end{cases}$$

In answering the above question, we prove the following results.

Theorem 1. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an increasing function such that $f(x) < x$ for all $x \in \mathbb{R}_{\geq 0}$. Suppose that there exists an x_0 such that, for each integer $n \geq x_0$, the interval $(n - f(n), n]$ contains at least one element of A. Then*

$$R_A(n) \ll t_{x_0}(n).$$

As an application of the above theorem, we have the following corollaries.

Corollary 1. *Suppose A satisfies the hypothesis of Theorem 1 with $f(x) = x^\theta$ for some $0 < \theta < 1$. Then*

$$R_A(n) \ll \log \log n,$$

where the implied constant may depend on θ.

Corollary 2. *Suppose A satisfies the hypothesis of Theorem 1 with $f(x) = \delta x$ for some $0 < \delta < 1$. Then*

$$R_A(n) \ll \log n,$$

where the implied constant may depend on δ.

Corollary 3. *For each real number $t \geq 1$, we let $f_t(x) = x/\log^t x$. If A satisfies the hypothesis of Theorem 1 with $f = f_t$ for each t, then*

$$R_A(n) = o(\log n).$$

The theorem below gives a lower bound for the function $R_A(n)$, and as an application of this theorem, we get Corollaries 4 and 5.

Theorem 2. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be an increasing function such that $f(x) < x$. Suppose there exists x_0 such that for all $x \geq x_0$, the interval $(1, x]$ contains a subinterval $I_{A,x}$ of length $f(x)$ such that $A \cap I_{A,x} = \emptyset$. Then for infinitely many positive integers n, we have*

$$R_A(n) \geq C t_{x_0}(n)$$

for some absolute constant C.

Corollary 4. *If A satisfies the hypothesis of Theorem 2 with $f(x) = x^\theta$ for some $\theta \in (0, 1)$, then*

$$R_A(n) \gg \log \log n$$

for infinitely many n.

Corollary 5. *If A satisfies the hypothesis of Theorem 2 with $f(x) = x/M$ for some $M > 12$, then*

$$R_A(n) \gg \log n$$

for infinitely many n.

In Sect. 3, we provide some examples to illustrate the above results.

2. Proofs. *Proof of Lemma 1.* Suppose we assume that

$$\limsup_{k\to\infty}(a_k - a_{k-1}) = \infty.$$

We have to show that for any integer $r \geq 1$, there is a n such that $R_A(n) \geq r$. Let m be a given integer. By our assumption, we can find an integer N sufficiently large such that

$$a_{N+1} - a_N > m + 1. \tag{2}$$

Let $n = m + a_N$. By (2), we conclude that $a(n) = a_N$. Therefore,

$$R_A(n) = R_A(m) + 1.$$

Now the result follows inductively.

Conversely, suppose $\limsup R_A(n) = \infty$. We want to prove that $\limsup_{k\to\infty}(a_k - a_{k-1}) = \infty$. Suppose not, that is, there exists a constant $M > 0$ such that $a_k - a_{k-1} \leq M$ for every k. Then, for any $n \in \mathbb{N}$, we have $n - a(n) \leq M$, so that $R_A(n) \leq M + 1$ for all n, which is a contradiction. □

Proof of Theorem 1. Let $n = n_1 \geq x_0$ be an integer. Then, by the assumption we have $a(n_1) \in (n_1 - f(n_1), n_1]$. Therefore $a(n_1) > n_1 - f(n_1)$. Thus the chain of inequalities

$$\begin{aligned} n_2 &= n_1 - a(n_1) < n_1 - n_1 + f(n_1) = f(n_1); \\ n_3 &= n_2 - a(n_2) < f(n_2) < f(f(n_1)) = f^2(n_1); \\ n_4 &< f^3(n_1); \\ \cdots & \quad \cdots \quad \cdots \\ n_{\ell+1} &< f^{\ell}(n_1) \end{aligned}$$

holds as long as $n_\ell \geq x_0$. We now let ℓ be the integer such that $n_{\ell+2} < x_0 \leq n_{\ell+1}$. We then have

$$f^{\ell}(n_1) \geq x_0.$$

Now, we put

$$b = \max_{1\leq m<n_0} R_A(m).$$

Then, $R_A(n_1) = R_A(n) \leq b + 1 + \ell$. Thus,

$$R_A(n_1) \leq Ct(n_1),$$

where C is a constant which may depend on x_0. □

Proof of Corollary 1. Given that $f(x) = x^{\theta}$ for some $\theta \in (0,1)$, we have $f^m(x) = x^{\theta^m}$ for any integer $m \geq 1$. By hypothesis, there exists x_0 such that for every integer $n \geq x_0$, the interval $(n - n^{\theta}, n]$ contains an element of A. Therefore, by Theorem 1, it is enough to find the function $t(n)$ as a function of x. For a given x, let ℓ be the largest positive integer such that $x_0 \leq x^{\theta^{\ell}}$. By applying logarithms on both sides, we get

$$\theta^{\ell}\log x \geq \log x_0,$$

which implies that

$$\ell \log \theta + \log \log x \geq \log \log x_0.$$

Since $\theta \in (0,1)$, we get

$$\ell \leq \frac{\log \log x - \log \log x_0}{\log(1/\theta)} \ll \log \log x.$$

□

Proof of Corollary 2. Let ℓ be the largest positive integer such that $x_0 \leq x\delta^\ell$. Therefore,

$$\ell \log \left(\frac{1}{\delta}\right) \leq \log x,$$

so that

$$\ell \leq \frac{\log x}{\log\left(\frac{1}{\delta}\right)}.$$

Hence, we get $\ell \ll \log x$. □

Proof of Corollary 3. Let t be fixed and n be a sufficiently large integer. We have

$$\begin{aligned}
n &= n_1; \\
n_2 &= n_1 - a(n_1) \leq n_1 - \left(n_1 - \frac{n_1}{(\log n_1)^t}\right) \leq \frac{n_1}{(\log n_1)^t}; \\
n_3 &\leq \frac{n_2}{(\log n_2)^t} \leq \frac{n_1}{(\log n_1)^t (\log n_2)^t} \leq \frac{n_1}{(\log n_2)^{2t}}; \\
\cdots &\quad \cdots \quad \cdots \\
n_{r+1} &\leq \frac{n}{(\log n_1)^t \cdots (\log n_r)^t} \leq \frac{n}{(\log n_r)^{rt}}.
\end{aligned}$$

Suppose r is the number of iterations, we need to get below x_0, then

$$x_0 \leq \frac{n}{(\log n_r)^{rt}}.$$

This implies

$$r \leq c\frac{\log n}{t}.$$

Since t is arbitrary, we get

$$R_A(n) = o(\log n).$$

□

Proof of Theorem 2. Let $x \geq x_0$ be a sufficiently large real number. By our assumption there exists $c(x) \in A$ such that the interval $(c(x), d(x)]$ does not contain any element of A and $d(x) - c(x) \geq f(x)$. Now applying the assumption to the interval $(1, d(x) - c(x)]$, there exists a $c_1(x) \in A$ such that $(c_1(x), d_1(x)]$

does not contain any element of A and $d_1(x) - c_1(x) \geq f(d(x) - c(x)) \geq f(f(x)) = f^2(x)$. Continue as above until we arrive at a stage

$$f^{t-1}(x) \geq x_0 \geq f^t(x).$$

Now let $n_0 = c(x) + c_1(x) + \cdots + c_{t-1}(x)$. We claim that

$$\begin{aligned} a(n_0) &= c(x); \\ a(n_1) &= c_1(x); \\ &\cdots \\ a(n_{t-1}) &= c_{t-1}(x). \end{aligned}$$

Hence, $R_A(n_0) = t_{x_0}(x) \geq t_{x_0}(n_0)$.

Now we will prove the above claim. Since $d_i(x) \leq d_{i-1}(x) - c_{i-1}(x)$, we have

$$\begin{aligned} c(x) \leq n_0 &\leq c(x) + c_1(x) + c_2(x) + \cdots + c_{t-2}(x) + d_{t-2} - c_{t-2}(x) \\ &\leq c(x) + c_1(x) + \cdots + c_{t-3}(x) + d_{t-3}(x) - c_{t-3}(x) \\ \cdots \quad &\cdots \quad \cdots \\ &\leq c(x) + d_1(x) \\ &\leq c(x) + d(x) - c(x) \end{aligned}$$

and since the interval $(c(x), d(x)]$ does not contain any element of A, we get that $a(n_0) = c(x)$. Similarly, one can prove the other equalities. Having constructed one integer n_0 with $R_A(n_0) \geq Ct(n_0)$, replacing x by sufficiently large numbers, we can construct infinitely many integers n satisfying $R_A(n) \geq Ct(n)$. This completes the proof of the theorem. □

Note: The proofs of the Corollary 4 and Corollary 5 are similar to that of Corollaries 1 and 2.

3. Examples.

(1) Let A be the set of all prime numbers.
  (a) By the Bertrand postulate, we know that every interval $\left(\frac{x}{2}, x\right]$ contains a prime number. Hence, by Corollary 2 with $f(x) = x/2$, we get $R_A(n) \ll \log n$.
  (b) By the prime number theorem, we know that every interval $\left(x - x/\log^t x, x\right]$ contains a prime number for all $x \geq x_0$ and for all $t \geq 0$. Hence, by Corollary 3, we have

$$R_A(n) = o(\log n), \qquad \text{as } n \to \infty.$$

  (c) Hoheisel [6] proved that there exist absolute constants $\theta \in (0, 1)$ and N_0 such that for every integer $n \geq N_0$, the interval $(n - n^\theta, n]$ contains a prime number. Therefore by Corollary 1, we get

$$R_A(n) \ll \log\log n.$$

  (d) Cramér's conjecture [4] asserts that for all $x > x_0$ the interval $[x, x + c\log^2 x]$ contains a prime number for some constant $c > 0$.

By assuming this conjecture and taking $f(x) = c\log^2 x$ in Theorem 1, we can arrive at $t(n) \ll \log_k n$ and hence we get

$$R_A(n) \ll \log_k n;$$

for every fixed positive integer k and $\log_k$ is k iterations of the log function.

(2) Let A be the set of all primes p of the form $m^2 + n^2 + 1$. In Wu [11], it is proved that every interval $(x, x + x^{115/121}]$ for all $x \geq x_0$ contains a prime of the form $p = m^2 + n^2 + 1$. Therefore, by Corollary 1, we get

$$R_A(n) \ll \log\log n.$$

(3) Let A be the set of all square-full numbers (n is called k-full number if $p \mid n$, then $p^k \mid n$) and put $S_A = \{1, 2, 3\}$. Then one can easily observe the following: for each $x \geq 1$, the interval $((\sqrt{x} - 1)^2, x]$ contains at least one square, namely, $[\sqrt{x}]^2$, we conclude that there is a constant $c_1 \geq 2.5$ such that for every $x \geq 1$, the interval $[x - c_1\sqrt{x}, x]$ contains a square-full number. So by Corollary 1, we have

$$R_A(n) \ll \log\log n.$$

Bateman and Grosswald [2] proved that for any $x \geq x_0$ the interval $[x/2, x]$ contains a subinterval I of length $x^{1/3}$ which contains no square-full number. Therefore, for infinitely many n

$$R_A(n) \geq C\log\log n.$$

If A is the set of all cube-full numbers, then by the result in [10], we get

$$R_A(n) \ll \log\log n.$$

(4) Let A be the set of all square-free numbers. Filaseta and Trifonov [5] proved that there exists a constant $C > 0$ such that for x sufficiently large the interval $(x, x + Cx^{1/5}\log x]$ contains a square-free number. Hence by Corollary 1, we get

$$R_A(n) \ll \log\log n.$$

(5) For a fixed $\delta > 0$, let

$$A = \left\{n \in \mathbb{N} : \mu(n) \neq 0, \quad \text{and} \quad \text{if } \ p|n, \text{ then } p \leq \exp(\log^{\delta} n)\right\}.$$

Then Charles [3] proved that every interval $[x, x + x^{1/2+\epsilon}]$ for $x \geq x_0$ and for every $\epsilon > 0$ contains an element of A if δ is very close to 1. Since A satisfies the hypothesis of Corollary 1, we have

$$R_A(n) \ll \log\log n.$$

(6) Let $\mathbb{B} = \{b_k\}_{k\geq 1}$ denote a sequence of integers satisfying $\sum \frac{1}{b_k} < \infty$ and $(b_i, b_j) = 1$ for $i \neq j$. Then the $\mathbb{B}$-free integers are those positive integers which are divisible by none of the b_k. Let $A = A_{\mathbb{B}}$ be the set of all $\mathbb{B}$-free numbers. In [12], Zhai proved that every interval $(x - x^{33/79}, x]$ contains a $\mathbb{B}$-free number for all sufficiently large x. By Corollary 1, we get

$$R_A(n) \ll \log\log n.$$

(7) A number n is called 'Deficient' if the sum of divisors of n is less than $2n$. $A = \{n : n \text{ is deficient}\}$. It is proved in [9] that there is always a deficient number between x and $x + \log^2 x$ for large enough x. Therefore, for any fixed k, we get

$$R_A(n) \ll \log_k n.$$

4. Champions. In this section, for each $A \subseteq \mathbb{N}$, we define a sequence of integers called champions and we study some of its properties.

Definition 2. Let $A \subseteq \mathbb{N}$. We say that an integer m is a A-champion if $R_A(n) < R_A(m)$ for all $n < m$.

The proposition below was first stated in Pillai [8] when A is the set of all prime numbers, and he did not provide its proof. Now we prove the general case as follows.

Proposition 1. *Let $A \subseteq \mathbb{N}$ and $\{t_r\}_{r\geq 1}$ be the sequence of A-champions. Then we have the following:*

(1) $t_1 = 1$.
(2) *(Recurrence formula)* $t_r = a(t_r) + t_{r-1}$.
(3) $R_A(t_r) = r$.

Proof. (1) Trivial.

(2) If $t_r - a(t_r) < t_{r-1}$, then

$$R_A(t_r - a(t_r)) < R_A(t_{r-1}) \text{ (Since } t_{r-1} \text{ is a champion).}$$

Hence

$$R_A(t_r) - 1 < R_A(t_{r-1}). \tag{3}$$

Since t_r is a champion and $t_{r-1} < t_r$, we have $R_A(t_{r-1}) + 1 \leq R_A(t_r)$, which gives a contradiction to (3). Hence

$$t_{r-1} + a(t_r) \leq t_r.$$

If $t_{r-1} + a(t_r) < t_r$, then since $R_A(t_{r-1}) < R_A(t_{r-1} + a(t_r)) < R_A(t_r)$ there will be a champion t such that $t_{r-1} < t < t_r$, a contradiction to the fact that t_r is the immediate successor of t_{r-1}. Hence the recurrence formula holds.

(3) By the above recurrence formula and by induction, the proof follows. □

The Proposition below asserts that the sequence t_r may grow very rapidly with r.

Proposition 2. *Let $A \subseteq \mathbb{N}$ and $\{t_r\}_{r\geq 1}$ be as in the above proposition.*

(1) *Suppose A satisfies the hypothesis of Theorem 1 with $f(x) = x^\theta$ for some $0 < \theta < 1$. Then $t_r \geq e^{e^{cr}}$ with some positive constant c.*
(2) *Suppose A satisfies the hypothesis of Theorem 1 with $f(x) = \delta x$ for some $0 < \delta < 1$. Then*

$$t_r \geq e^{cr}$$

with some positive constant c.

Proof. (1) If A is as in the hypothesis, then by Corollary 1, we have that

$$R_A(n) \ll \log\log n.$$

By (3) of the above proposition, we have $R_A(t_r) = r$. Hence

$$r = R_A(t_r) \ll \log\log t_r,$$

by which we get that

$$t_r \geq e^{e^{cr}}.$$

(2) The proof is similar to the one above. □

Example 2. Let A be set of all prime numbers. In [1], it is proved that any interval $(x - x^{0.525}, x]$ contains a prime number for sufficiently large x. From this we can deduce that $R_A(n) \leq 2\log\log n$ for sufficiently large n. Hence we see that in this case the sequence satisfies

$$t_r \geq e^{e^{r/2}}$$

for sufficiently large r.

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