

ON SHORT ZERO-SUM SEQUENCES OVER ABELIAN P -GROUPS

M. N. Chintamani

*School of Mathematics and Statistics, University of Hyderabad, Hyderabad 500046,
India*

mohansspecial@gmail.com

Prabal Paul

*Department of Mathematics, BITS-Pillani, K.K. Birla Goa Campus, Goa 403726,
India*

prabalpaul@goa.bits-pilani.ac.in

R. Thangadurai

*Harish-Chandra Research Institute (HBNI), Chhatnag Road, Jhansi, Allahabad
211019, India*

thanga@hri.res.in

Received: , Revised: , Accepted: , Published:

Abstract

Let G be a finite abelian additive group. In this paper, we deal with a combinatorial constant related to short zero-sum sequences over the abelian p -groups.

1. Introduction

Let G be a finite abelian additive group with exponent $\exp(G)$. A sequence S over G is written as

$$S = \prod_{i=1}^{|S|} g_i = \prod_{g \in G} g^{v_g(S)} \text{ with } v_g(S) \in \mathbb{Z}_{\geq 0}$$

where $v_g(S)$ is called the *multiplicity* of g in S and $|S|$ denotes the length of the sequence S . By the definition of multiplicity, we see that

$$|S| = \sum_{g \in G} v_g(S) \in \mathbb{Z}_{\geq 0}.$$

The sum of all the terms of the sequence S is given by

$$\sigma(S) = \sum_{g \in G} v_g(S)g \in G.$$

A sequence S over G is called a *zero-sum sequence* if $\sigma(S) = 0$. A sequence S is called a *short zero-sum sequence* if $\sigma(S) = 0$ and $|S| \in [1, \exp(G)]$. For any integer $k \in \mathbb{Z}_{\geq 0}$ and for a sequence S over G , we define

$$N^k(S) = \left| \left\{ I \subset [1, |S|] : \sum_{i \in I} g_i = 0, |I| = k \right\} \right|,$$

which denotes the number of zero-sum subsequences of S of length k . The Davenport constant, $D(G)$, is the minimal positive integer t such that any given sequence S over G of length $|S| \geq t$ contains a nonempty zero-sum subsequence. The constant $\eta(G)$ is the minimal positive integer t such that any given sequence S over G of length $|S| \geq t$ contains a short zero-sum subsequence. Finally, the EGZ constant $s(G)$ is the minimal positive integer t such that any given sequence S over G of length $|S| \geq t$ contains a zero-sum subsequence T of length $|T| = \exp(G)$.

These constants are classical invariants attached to a finite abelian group G in combinatorial number theory and have received a lot of attention (see for instance [1, 2, 5, 6, 7, 9, 10, 15, 16]). When G is a cyclic group, we have $\eta(G) = |G|$ and $s(G) = 2|G| - 1$, by the well-known Erdős-Ginzburg-Ziv theorem [4]. For this contribution, this constant $s(G)$ is called *EGZ constant*. When $G \cong C_p^2$ for a prime p , Olson [13, 14] proved in 1969 that $\eta(C_p^2) = 3p - 2$ and C. Reiher [15] proved in 2007 that $s(C_p^2) = 4p - 3$, which was, earlier, conjectured by Kemnitz [11] in 1983. In general, if $G \cong C_m \oplus C_n$ with $m|n$ is the abelian group of rank 2, then it is known that $s(G) = \eta(G) + n - 1 = 2m + 2n - 3$ as given in [10]. In 1995, Alon and Dubiner [1] proved that $s(C_n^r) \leq c(r)n$ where $c(r)$ is a computable constant depends only on the rank r .

When G is of rank ≥ 3 , nothing more is known. Even when $G \cong C_p^3$, for any prime p , these constants are still unknown. Recently, Fan, Gao, Wang and Zhong [7] determined the values $\eta(G)$ and $s(G)$ for special type of abelian groups of rank 3. Apart from these results, Schmid and Zhuang [16] proved that if G is a finite abelian p -group with $D(G) = 2 \exp(G) - 1$, then $s(G) = 2D(G) - 1 = \eta(G) + \exp(G) - 1$. Moreover, they conjectured the following.

Conjecture 1. ([16]) *Let G be a finite abelian p -group with $D(G) \leq 2 \exp(G) - 1$. Then*

$$s(G) = 2D(G) - 1 = \eta(G) + \exp(G) - 1.$$

In this article, we prove the following theorems toward Conjecture 1 for a large class of abelian p -groups using the techniques employed in a recent paper of Gao, Han and Zhang [8].

Theorem 1. *Let H be a finite abelian p -group of rank $r(H)$ and $\exp(H) = p^m$ for some positive integer m and for some prime p with $p > 2r(H)$ and $D(H) - 1 =$*

$kp^m + t$ for some positive integer k and a non-negative integer t satisfying $0 \leq t \leq (p^m - 1)/2$. For any positive integer n with $p^n \geq 2(D(H) - 1)$, let $G = C_{p^n} \oplus H$ be the abelian p -group satisfying $D(G) \leq 2p^n - 1 = 2 \exp(G) - 1$. Let S be a sequence over G of length $p^n + 2(D(H) - 1)$. If $N^{p^n + j_0 p^m}(S) = 0$ for some integer j_0 with $1 \leq j_0 \leq k$, then S contains a short zero-sum subsequence.

In [8], Gao, Han and Zhang proved the Conjecture 1 for the abelian p -groups G satisfying $k = 1$ or $t \in [p^m/2, p^m)$ (notation is as in Theorem 1). In the following theorem, we deal with the complement of this result.

Theorem 2. *Let H be a finite abelian p -group of rank $r(H)$ and $\exp(H) = p^m$ for some positive integer m and for some prime p with $p > 2r(H)$ and $D(H) - 1 = kp^m + t$ for some positive integer k and a non-negative integer t satisfying $0 \leq t \leq (p^m - 1)/2$. For any positive integer n with $p^n \geq 2(D(H) - 1) + p^m$, let $G = C_{p^n} \oplus H$ be the abelian p -group satisfying $D(G) \leq 2p^n - 1 = 2 \exp(G) - 1$. Then, we have,*

$$\eta(G) \leq 2D(G) - \exp(G) + (\exp(H) - t - 1) = p^n + 2(D(H) - 1) + p^m - t.$$

Note that when $G = C_{p^n} \oplus H$, then $D(G) = p^n + D(H) - 1$. Therefore, Theorem 2 states that $\eta(G) \leq (2D(G) - 1) - \exp(G) + (\exp(H) - t)$ for the case $t \in [0, (p^m - 1)/2]$ and hence $\exp(H) - t - 1$ is the extra term againsts the Conjecture 1.

2. Preliminaries

Throughout this section, we take H to be a finite abelian p -group of rank $r(H)$ and exponent $\exp(H) = p^m$ for some positive integer m . Also, we write $D(H) - 1 = kp^m + t$ for some positive integer k and a non-negative integer t satisfying $0 \leq t \leq (p^m - 1)/2$. Choose any integer n such that $p^n \geq 2(D(H) - 1)$ and let $G = C_{p^n} \oplus H$.

We have the following lemmas which are needed in the proof of Theorem 1 and Theorem 2.

Lemma 2.1. ([8]) *Let $v = (k + 1)p^m - D(H) = p^m - t - 1$. Let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1)$ such that S contains no short zero-sum subsequences. For any integer i with $0 \leq i \leq k - 1$, let T be a subsequence of S of length $|T| = |S| - ip^m$. Then we have the following;*

$$1 + \sum_{u=0}^h \binom{h}{u} \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p}, \tag{2.1}$$

for all $h \in [0, v]$.

Lemma 2.2. *Let $v = (k+1)p^m - D(H) = p^m - t - 1$. Let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1)$ such that S contains no short zero-sum subsequences. For any integers i and h satisfying $0 \leq i \leq k - 1$ and $0 \leq h \leq v - 1$, we have*

$$\binom{|S|}{ip^m} + \sum_{j=1}^k (-1)^{j-1} \sum_{u=0}^h \binom{h}{u} \binom{|S| - p^n - jp^m + u}{ip^m} N^{p^n + jp^m - u}(S) \equiv 0 \pmod{p}. \tag{2.2}$$

Proof. This lemma is implicitly proved in Lemma 3.1 (3.3) of [8]. In order to get (2.2), we take a subsequence T of S such that $|T| = |S| - ip^m$ for a given integer i with $0 \leq i \leq k - 1$. We can get

$$1 + \sum_{u=0}^h \binom{h}{u} \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p}.$$

Now we sum over all the subsequences T with $|T| = |S| - ip^m$ and we get

$$\sum_{T, |T|=|S|-ip^m} \left(1 + \sum_{u=0}^h \binom{h}{u} \sum_{j=1}^k (-1)^{j-1} N^{p^n + jp^m - u}(T) \right) \equiv 0 \pmod{p}. \tag{2.3}$$

Since each subsequence W of S with $|W| \leq |S| - ip^m$ can be extended to a subsequence T of length $|T| = |S| - ip^m$ in

$$\binom{|S| - |W|}{|T| - |W|} = \binom{|S| - |W|}{|S| - |T|} = \binom{|S| - |W|}{ip^m}$$

ways, by starting with 0 length subsequence W of S , we see that the number of ways to get subsequences T of S with $|T| = |S| - ip^m$ is $\binom{|S|}{ip^m}$. Then, using this and expanding the sum in (2.3), we arrive at (2.2). \square

Corollary 2.2.1. *Let S be a sequence over G as defined in Lemma 2.2. For any integer i with $0 \leq i \leq k - 1$, we have*

$$\binom{|S|}{ip^m} + \sum_{j=1}^k (-1)^{j-1} \binom{|S| - p^n - jp^m}{ip^m} N^{p^n + jp^m}(S) \equiv 0 \pmod{p}. \tag{2.4}$$

Proof. Put $h = 0$ in Lemma 2.2 to get the result. \square \square

Theorem 2.3. ([12]) *Let p be a prime number. Let a and b be positive integers with $a = a_n p^n + a_{n-1} p^{n-1} + \dots + a_0$ with $a_i \in \{0, 1, \dots, p-1\}$ and $b = b_n p^n + b_{n-1} p^{n-1} + \dots + b_0$ with $b_i \in \{0, 1, \dots, p-1\}$. Then*

$$\binom{a}{b} \equiv \binom{a_n}{b_n} \binom{a_{n-1}}{b_{n-1}} \dots \binom{a_0}{b_0} \pmod{p}.$$

Theorem 2.4. ([8]) *Let n and k be positive integers with $1 \leq 2k \leq n$. Let A be the following $(k+1) \times (k+1)$ matrix with positive integers*

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \binom{n}{1} & \binom{n-1}{1} & \dots & \binom{n-k}{1} \\ \binom{n}{2} & \binom{n-1}{2} & \dots & \binom{n-k}{2} \\ \dots & \dots & \dots & \dots \\ \binom{n}{k} & \binom{n-1}{k} & \dots & \binom{n-k}{k} \end{pmatrix}.$$

Then, the determinant of A is

$$\det(A) = \left(\prod_{t=1}^k t! \right)^{-1} \prod_{1 \leq i < j \leq k} (j - i).$$

3. Proof of Theorem 1

Proof of Theorem 1. Let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1)$. By the assumption, for some integer j_0 with $0 \leq j_0 \leq k$, we have $N^{p^n + j_0 p^m}(S) = 0$. Without loss of generality, we shall assume that $j_0 = k$ and hence $N^{p^n + k p^m}(S) = 0$, as the proof of the other cases are similar. We need to prove that S contains a short zero-sum subsequence. On the contrary, we shall assume that S contains no short zero-sum subsequences. Hence, by Corollary 2.2.1, for any integer i with $0 \leq i \leq k - 1$ and by the assumption with $j_0 = k$, we get

$$\binom{|S|}{i p^m} + \sum_{j=1}^{k-1} (-1)^{j-1} \binom{|S| - p^n - j p^m}{i p^m} N^{p^n + j p^m}(S) \equiv 0 \pmod{p}. \tag{3.1}$$

Note that for any integer j with $1 \leq j \leq k - 1$, we have

$$|S| - p^n - j p^m = p^n + 2(k p^m + t) - p^n - j p^m = (2k - j) p^m + 2t.$$

Since $D(H) - 1 = k p^m + t$ for some integer t with $0 \leq t \leq (p^m - 1)/2$ and $p > 2r(H)$, we see that

$$D(H) - 1 \leq r(H) \exp(H) - r(H) < r(H) p^m \leq \left(\frac{p}{2} - 1 \right) p^m \implies k \leq \frac{p}{2} - 1.$$

Therefore, for all integers j with $1 \leq j \leq k - 1$, we see that $2k - j < 2k < p$ and every integer $i < k < p$. Since $2t \leq p^m - 1 < p^m$, by Theorem 2.3, we get

$$\binom{|S| - p^n - jp^m}{ip^m} \equiv \binom{2k - j}{i} \binom{2t}{0} \equiv \binom{2k - j}{i} \pmod{p} \tag{3.2}$$

for all integers j with $1 \leq j \leq k - 1$ and for all integers i with $0 \leq i \leq k - 1$. Also, since $|S| = p^n + 2kp^m + 2t$ and $2t < p^m < p^n$, by Theorem 2.3, we get

$$\binom{|S|}{ip^m} \equiv \binom{2k}{i} \pmod{p} \tag{3.3}$$

for all integers i with $0 \leq i \leq k - 1$. Therefore, by (3.1), (3.2) and (3.3), we get

$$\binom{2k}{i} + \sum_{j=1}^{k-1} \binom{2k - j}{i} (-1)^{j-1} N^{p^n + jp^m}(S) \equiv 0 \pmod{p} \tag{3.4}$$

for all $i = 0, 1, \dots, k - 1$. Now, put

$$X_j = (-1)^{j-1} N^{p^n + jp^m}(S)$$

for all $j = 1, 2, \dots, k - 1$ and $X_0 = 1$ as variables modulo p . Then, by putting $i = 0, 1, 2, \dots, k - 1$ in (3.4), we get a system of k linear equations in k variables modulo p as follows.

$$\begin{aligned} X_0 + X_1 + X_2 + \dots + X_{k-1} &= 0; \\ \binom{2k}{1} X_0 + \binom{2k - 1}{1} X_1 + \binom{2k - 2}{1} X_2 + \dots + \binom{k + 1}{1} X_{k-1} &= 0; \\ &\dots\dots\dots \\ &\dots\dots\dots \\ \binom{2k}{k - 1} X_0 + \binom{2k - 1}{k - 1} X_1 + \binom{2k - 2}{k - 1} X_2 + \dots + \binom{k + 1}{k - 1} X_{k-1} &= 0; \end{aligned}$$

Note that the coefficient matrix of the above system of linear equations is nothing but A in Theorem 2.4 with $n = 2k$ and $k = k - 1$. Therefore, by Theorem 2.4, the determinant of the coefficient matrix is non-zero modulo p which forces the system to have only the trivial solution modulo p . That is,

$$X_0 \equiv X_1 \equiv \dots \equiv X_{k-1} \equiv 0 \pmod{p},$$

which is a contradiction to $X_0 = 1 \not\equiv 0 \pmod{p}$. This proves the result. □
□

4. Proof of Theorem 2

To prove Theorem 2, first we need to extend Lemma 2.2 so that it holds true for all $i \in [0, k]$ and for all $h \in [0, v)$, when $|S| = p^n + 2(D(H) - 1) + p^m - t$ for some integer t satisfying $0 \leq t \leq (p^m - 1)/2$ (notation is as in Section 2). Recall that H is an abelian p -group of rank $r(H)$ and exponent p^m . Let n be a positive integer such that $p^n \geq 2(D(H) - 1) + p^m$ and $G = C_{p^n} \oplus H$.

Lemma 4.1. *Let S be a given sequence over G of length $|S| = p^n + 2(D(H) - 1) + p^m - t$ where t is an integer satisfying $D(H) - 1 = kp^m + t$ with $0 \leq t \leq (p^m - 1)/2$. Let $v = (k+1)p^m - D(H) = p^m - t - 1$. If S contains no short zero-sum subsequences, then for all integers h and i satisfying $0 \leq h < v$ and $0 \leq i \leq k$, we have*

$$\binom{|S|}{ip^m} + \sum_{j=1}^k (-1)^{j-1} \sum_{u=0}^h \binom{h}{u} \binom{|S| - p^n - jp^m + u}{ip^m} N^{p^n + jp^m - u}(S) \equiv 0 \pmod{p}. \tag{4.1}$$

Proof. The proof of this lemma is similar to the proof of Lemma 3.1 of [8]. Let S be the given sequence over G of length $|S| = p^n + 2(D(H) - 1) + p^m - t$ and S contains no short zero-sum subsequences.

Claim 1. $N^a(S) = 0$ for all integers a satisfying $1 \leq a \leq p^n$ or $p^n + D(H) \leq a \leq |S|$.

Since S contains no short zero-sum subsequences, we see that $N^a(S) = 0$ for all integers a satisfying $1 \leq a \leq p^n$. By the argument of Lemma 3.1 of [8], we see that $N^a(S) = 0$ for all integers a satisfying $p^n + D(H) \leq a \leq |S| - p^m + t$. Thus, to prove Claim 1, we need to prove that $N^a(S) = 0$ for all integers a satisfying $|S| - p^m + t + 1 \leq a \leq |S|$.

Let W be a zero-sum subsequence of S of length $|W| = a$ for some integer a satisfying $|S| - p^m + t + 1 \leq a \leq |S|$. Since $D(G) = p^n + D(H) - 1$ (by [13]) and $|W| = a \geq |S| - p^m + t + 1 = p^n + 2(D(H) - 1) + 1$, we see that W contains at least two disjoint zero-sum subsequences W_1 and W_2 such that $W = W_1W_2$. Since $N^b(S) = 0$ for all integers $1 \leq b \leq p^n$, we see that $|W_1| \geq p^n + 1$ and $|W_2| \geq p^n + 1$ and hence $|W| \geq 2p^n + 2$, which is a contradiction because $|S| < 2p^n + 2$ (as by hypothesis, $p^n \geq 2(D(H) - 1) + p^m$). This proves Claim 1.

Using Claim 1, the proof of Lemma 3.1 (3.1) of [8] yields Lemma 2.1 in Section 2 for the sequence S which in turn produces the congruence (4.1) for all integers i and h satisfying $0 \leq i \leq k - 1$ and $0 \leq h < v$. Hence, it is enough to prove the congruence (4.1) for $i = k$ and for all integers h with $0 \leq h < v$. Let T be any subsequence of S of length $|T| = |S| - kp^m$. Then consider the sequence $T0^h$ for a

given integer h with $0 \leq h < v$. Note that

$$\begin{aligned} |T0^h| &= |T| + h = |S| - kp^m + h \\ &= p^n + 2(D(H) - 1) + p^m - t - kp^m + h \\ &= p^n + D(H) - 1 + p^m + D(H) - 1 - kp^m - t + h \\ &\geq D(G) + p^m - 1. \end{aligned}$$

Then the rest of the proof is just same as that of Lemma 3.1 in [8]. □

Proof of Theorem 2. Let H be a finite abelian p -group of rank $r = r(H)$ and the exponent p^m . Let $D(H) - 1 = kp^m + t$ for some positive integer k and for some integer t with $0 \leq t \leq (p^m - 1)/2$. Let n be an integer such that $p^n \geq 2(D(H) - 1) + p^m$ and let $G = C_{p^n} \oplus H$.

To prove $\eta(G) \leq p^n + 2(D(H) - 1) + p^m - t$, we let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1) + p^m - t$ and we prove that S contains a short zero-sum subsequence.

Suppose S contains no short zero-sum subsequences. Hence, by Lemma 4.1 with $h = 0$, we get

$$\binom{|S|}{ip^m} + \sum_{j=1}^k (-1)^{j-1} \binom{|S| - p^n - jp^m}{ip^m} N^{p^n + jp^m}(S) \equiv 0 \pmod{p}, \quad (4.2)$$

for all integers i with $0 \leq i \leq k$. Note that for all integers j satisfying $0 \leq j \leq k$, we have

$$|S| - p^n - jp^m = p^n + 2(kp^m + t) + p^m - t - p^n - jp^m = (2k + 1 - j)p^m + t.$$

Since $D(H) - 1 = kp^m + t$ for some integer t with $0 \leq t \leq (p^m - 1)/2$ and $p > 2r(H)$, we see that

$$D(H) - 1 \leq r(H) \exp(H) - r(H) < r(H)p^m \leq \left(\frac{p}{2} - 1\right)p^m \implies k \leq \frac{p}{2} - 1 \iff 2k + 2 \leq p.$$

Therefore, for all integers j with $1 \leq j \leq k$, we see that $2k + 1 - j < 2k + 1 < p$ and every integer $i \leq k < p/2$. Also, since $|S| = p^n + 2(D(H) - 1) + p^m - t = p^n + 2(kp^m + t) + p^m - t = p^n + (2k + 1)p^m + t$, and $2t \leq p^m - 1 < p^m$, by Theorem 2.3, we get

$$\binom{|S| - p^n - jp^m}{ip^m} \equiv \binom{2k + 1 - j}{i} \binom{t}{0} \equiv \binom{2k + 1 - j}{i} \pmod{p}$$

for all integers j satisfying $1 \leq j \leq k$. Also, since $|S| = p^n + (2k + 1)p^m + t$ and $2t < p^m < p^n$, by Theorem 2.3, we get

$$\binom{|S|}{ip^m} \equiv \binom{2k + 1}{i} \pmod{p}$$

for all integers i with $0 \leq i \leq k$. Therefore, by (4.2), we get

$$\binom{2k + 1}{i} + \sum_{j=1}^k \binom{2k + 1 - j}{i} (-1)^{j-1} N^{p^n + jp^m}(S) \equiv 0 \pmod{p} \tag{4.3}$$

for all integers i satisfying $i = 0, 1, \dots, k$.

Now, put

$$X_j = (-1)^{j-1} N^{p^n + jp^m}(S)$$

for all $j = 1, 2, \dots, k$ and $X_0 = 1$ as variables modulo p . By (4.3), we have the following system of linear equations in $k + 1$ variables modulo p .

$$\begin{aligned} X_0 + X_1 + X_2 + \dots + X_k &= 0; \\ \binom{2k + 1}{1} X_0 + \binom{2k + 1 - 1}{1} X_1 + \binom{2k + 1 - 2}{1} X_2 + \dots + \binom{2k + 1 - k}{1} X_k &= 0; \\ &\dots\dots\dots \\ &\dots\dots\dots \\ \binom{2k + 1}{k} X_0 + \binom{2k + 1 - 1}{k} X_1 + \binom{2k + 1 - 2}{k} X_2 + \dots + \binom{2k + 1 - k}{k} X_k &= 0; \end{aligned}$$

Note that the coefficient matrix of the above system of linear equations is nothing but A in Theorem 2.4 with $n = 2k + 1$ and $k = k$. Therefore, by Theorem 2.4, the determinant of the coefficient matrix is non-zero modulo p which forces the system to have only the trivial solution modulo p . That is,

$$X_0 \equiv X_1 \equiv \dots \equiv X_k \equiv 0 \pmod{p},$$

which is a contradiction as $X_0 = 1 \not\equiv 0 \pmod{p}$. This proves the theorem. □

Acknowledgement. We thank the referee for carefully going through the earlier version of the manuscript and gave useful suggestions to improve the presentation.

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