On a Problem of Alaoglu and Erdős

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Starting with an elementary problem that appeared in the Putnam mathematics competition, we proceed to discuss some techniques of transcendental number theory and prove the following result. If p, q, r are distinct primes and if c is a real number with the property that p^c, q^c, r^c are integers, then c must be a non-negative integer. The tools used are some linear algebra and complex analysis. The zero-density estimate method discussed here was used by Alan Baker to prove his celebrated theorem on linear forms in logarithms. The question as to whether we can replace three primes by two primes is an open question.

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1. A Putnam Problem

For all natural numbers n, evidently n^c is an integer if c is any non-negative integer. An obvious question is whether the converse also holds true. This was a question in one of the Putnam competitions and can be answered affirmatively as follows.

Putnam Problem. *If n^c is an integer for all natural numbers n, then is it true that c must be a non-negative integer?*

The proof will use forward differences defined for any function f by:

$$(\Delta f)(x) = f(x+1) - f(x).$$

We may define $\Delta^k f = \Delta(\Delta^{k-1} f)$ for any $k > 1$. It is easy to show by induction on k that

$$(\Delta^k f)(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x+j).$$

Now, since 2^c is an integer, we conclude that c must be a non-negative real number. If $0 < c < 1$, consider the function

$f(x) = x^c$ on the interval $[n, n+1]$ for any integer n satisfying $n > c^{1/(1-c)}$. Then, by the mean value theorem, there exists a $\xi \in (n, n+1)$ satisfying

Keywords
Six exponentials theorem, Siegel's lemma, zero-density estimate, transcendental method.

$$c\xi^{c-1} = (n+1)^c - n^c \in \mathbb{N}.$$

The choice of n implies that $c\xi^{c-1} < 1$, which is a contradiction. Therefore, we may assume that $c > 1$, because if $c = 1$, there is nothing to prove.

Since $c > 1$, there exists a unique integer k such that $k-1 \leq c < k$. Note that the function $f(x) = x^c$ is differentiable of order k on $[n, n+k]$ for every integer $n \geq 1$. By the generalized mean value theorem, there exists a $\xi \in (n, n+k)$ satisfying

$$f^{(k)}(\xi) = (\Delta^k f)(n),$$

where

$$(\Delta^k f)(n) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(n+j).$$

Since $(n+j)^c$ is an integer for all integers $j \geq 0$, we see that $(\Delta^k f)(n)$ is an integer. Since

$$f^{(k)}(\xi) = c(c-1)\ldots(c-k+1)\xi^{c-k} \geq 0,$$

we see that $(\Delta^k f)(n)$ is a non-negative integer. Also, since $c < k$ and $\xi \in (n, n+k)$, we get

$$\xi^{c-k} = \frac{1}{\xi^{k-c}} < \frac{1}{n^{k-c}}.$$

If n^c is an integer for all natural numbers n, then is it true that c must be a non-negative integer?

Since m^c is integer for every integer $m \geq 1$, we can choose n as large as possible such that

$$f^{(k)}(\xi) < \frac{c(c-1)\ldots(c-k+1)}{n^{k-c}} < 1$$

and hence we get, $c(c-1)\ldots(c-k+1) = 0$. Since $k-1 \leq c < k$, we get $c = k-1$.

The above solution immediately shows that the following modified finite version of the problem has an affirmative answer.

Modified Putnam Problem. *Let $c \geq 1$ be a given positive real number and set*

$$M = ([c] + 1)! + [c] + 1,$$

where $[c]$ is the integral part of c. If n^c is an integer for all natural numbers n satisfying $1 \leq n \leq M$, then c must be an integer.

At this juncture, one may ask the following natural question.

Question. *Let c be a given positive real number and let M be the integer as defined above. Let S be a non-empty subset of $\{1, 2, \ldots, M\}$. If n^c is an integer for all $n \in S$, can we conclude that c is an integer?*

We take $c = (\log 3)/(\log 2)$ and consider the corresponding M as above. Then the set $S = \{2, 2^2, \ldots, 2^\ell\}$ for some natural number ℓ with $2^\ell \leq M$ is a subset of $\{1, 2, \ldots, M\}$. Also, note that for any $n = 2^k \in S$ with $1 \leq k \leq \ell$, we have

$$n^c = (2^k)^c = 2^{(k \log 3)/(\log 2)} = 3^k$$

is an integer. However, $c = (\log 3)/(\log 2)$ is not an integer because $2^r \neq 3$ for any integer r. Thus the above question is not true for any non-empty subset S of $\{1, 2, \ldots, M\}$.

We shall observe the above example more closely. First note that for all singleton subsets of $\{1, 2, \ldots, M\}$, the above question is not true (by taking $\ell = 1$ in the above example). Moreover, any two elements in S are multiplicatively dependent[1]. The right question to be asked may be the following.

[1]We recall that two integers a and b are said to be *multiplicatively dependent*, if there exist integers x and y with $(x, y) \neq (0, 0)$ such that $a^x b^y = 1$; otherwise, they are called *multiplicatively independent.*

Modified Question. *Let c be a given positive real number and let M be the integer as defined above. Let S be a subset of $\{1, 2, \ldots, M\}$ satisfying $|S| \geq 2$ and any two elements of S are multiplicatively independent. If n^c is an integer for all $n \in S$, then can we conclude that c is an integer?*

In 1966/67, K Ramachandra [1] and S Lang [2] (independently) answered the above question in a more general setup (so called the 'six exponentials theorem'). In particular, one can deduce the following statement. *Let S be a subset of natural numbers* *such that $|S| \geq 3$ and any two elements of S are multiplicatively independent. If n^c is an integer for all $n \in S$ for some non-zero real number c, then c must be an integer.*

Let p and q be two distinct prime numbers. If p^c and q^c are integers for some non-zero real number c, can we conclude that c is an integer?

In 1944, L Alaoglu and P Erdős [3] asked the following optimal question.

Alaoglu–Erdős Problem. *Let p and q be two distinct prime numbers. If p^c and q^c are integers for some non-zero real number c, can we conclude that c is an integer?*

Indeed, it is expected that *for any two multiplicatively independent integers a and b, if a^c and b^c are integers for some non-zero real number c, then c must be a non-negative integer*.

The Alaoglu–Erdős problem is not yet solved till today even for a particular pair of distinct prime numbers. In this discussion, we shall prove a particular case of Lang–Ramachandra's theorem. The proof runs through the same path as the 'six exponentials theorem' proved by S Lang and K Ramachandra, with much less complications.

Theorem 1. *If $2^c, 3^c$ and 5^c are integers for some non-zero real number c, then c must be integer.*

Note that in the statement of Theorem 1, we can replace the primes $2, 3$ and 5 by any three distinct primes p, q and r and the proof is valid verbatim, except for changing some explicit constants.

2. Preliminaries

We recall some basic facts from linear algebra and complex analysis which are needed to prove Theorem 1.

§2.1. Linear Algebra – Siegel's Lemma. In the undergraduate course, when we solve a system of homogenous linear equations in n-variables, we learnt that *for a given system of homogeneous linear equations with integer coefficients in n unknowns and m equations, if $n > m$, then we have infinitely many n-tuples with integer co-ordinates that are simultaneous solutions of the system.*

The following lemma is due to C L Siegel which deals with an upper bound for the least such solution in terms of the coefficients of the system of equations.

Lemma 2.1. (Siegel [2] and [4], p.213) *Let n and m be given integers such that* $1 < m < n$. *Suppose*

$$a_{11}X_1 + a_{12}X_2 + \cdots + a_{1n}X_n = 0;$$

$$a_{21}X_1 + a_{22}X_2 + \cdots + a_{2n}X_n = 0;$$

$$\cdots\cdots\cdots$$

$$a_{m1}X_1 + a_{m2}X_2 + \cdots + a_{mn}X_n = 0;$$

with $a_{ij} \in \mathbb{Z}$ *for all* $1 \leq i \leq m$ *and* $1 \leq j \leq n$ *is a system of homogenous linear equations with integer coefficients. Suppose* $|a_{ij}| \leq A$ *for some real number* $A > 0$. *Then there exists* $(y_1, y_2, \ldots, y_n) \in \mathbb{Z}^n \backslash \{(0, 0, \ldots, 0)\}$ *such that*

$$|y_i| \leq (2nA)^{m/(n-m)} \textit{ for all } i = 1, 2, \ldots, n.$$

The proof follows by a clever application of the pigeonhole principle.

§2.2. Complex Analysis Tools. We recall some complex analysis results which will be useful to prove Theorem 1.

(2.2.1) Identity Theorem [5]**.** *Let f be an entire function on the complex plane* $\mathbb{C}$. *Suppose a sequence* $(z_n)_n$ *in* $\mathbb{C}$ *converges to some point* $z_0 \in \mathbb{C}$ *and* $f(z_n) = 0$ *for all* $(z_n)_{n\geq 1}$. *Then f is the zero function.*

(2.2.2) If f is an entire function on $\mathbb{C}$ which has 'lots of zeros' inside a disc D centered at the origin and radius $R > 0$. Then the following Lemma asserts that the functional value of f inside the disc D is 'relatively' very small. Sometimes it is also known as 'small value estimates'.

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Schwarz Lemma [5]**.** *Let* $R > 0$ *be any real number and* $N \geq 0$ *be any integer. Let f be an analytic function in a disc* $|z| \leq R$ *in* $\mathbb{C}$. *For any real number k with* $0 < k < R$, *we assume that f has* *at least N zeros in the disc $|z| \leq k$. Then*

$$\|f\|_k \leq \left(\frac{3k}{R}\right)^N \|f\|_R,$$

where

$$\|f\|_k = \sup_{|z|=k} |f(z)|.$$

(2.2.3) Let x_1, x_2 and x_3 be complex numbers such that they are linearly independent over $\mathbb{Q}$. Then for any non-zero polynomial $P(x, y, z) \in \mathbb{Z}[x, y, z]$, the entire function $P(e^{x_1 z}, e^{x_2 z}, e^{x_3 z})$ is not identically zero function. In other words, we say the functions $e^{x_1 z}, e^{x_2 z}$ and $e^{x_3 z}$ are algebraically independent as functions over $\mathbb{C}$ (see [6]).

(2.2.4) Let α be an irrational number. Then the set $S = \{a\alpha + b : a, b \in \mathbb{Z}\}$ is dense in $\mathbb{R}$ (see [7]).

3. Proof of Theorem 1

We prove Theorem 1 by a contradiction method. That is, by assuming c is not an integer, we get a contradiction. In order to get a contradiction, we shall use a 'transcendental method'. The method allows one to construct a non-zero integer, say, m_0 (related to the given inputs) and hence $|m_0| \geq 1$. Using the complex analysis tools, we prove that $|m_0| < 1$, which is a contradiction and finishes the proof.

In order to construct such an integer, we create a system of linear equations with integer coefficients where the number of variables is strictly more than the number of equations. Then, the linear algebra asserts that there are infinitely many integer solutions to the system. We use one of the first non-zero integer solutions to construct such an integer m_0.

Let n be a very large integer, which acts as a parameter and choose an integer r whose magnitude is roughly like $2n^{1/3}$. Then we can find integers $a_{uvw} \in \mathbb{Z}$, not all zeros, and an entire function

$$F(z) = \sum_{u=1}^{r} \sum_{v=1}^{r} \sum_{w=1}^{r} a_{uvw} 2^{uz} 3^{vz} 5^{wz}. \qquad (3.1)$$

In fact, we arrive at this entire function by creating a system of homogeneous linear equations with integer coefficients as follows. For any given integers a and b with $1 \leq a, b \leq \sqrt{n}$, we consider the following homogeneous linear equation

$$\sum_{u=1}^{r} \sum_{v=1}^{r} \sum_{w=1}^{r} 2^{(ac+b)u} 3^{(ac+b)v} 5^{(ac+b)w} X_{uvw} = 0. \qquad (3.2)$$

Since, by hypothesis, $2^c, 3^c$ and 5^c are integers, we see that the above system of homogeneous linear equations has integer coefficients. Further, each coefficient is bounded above by $30^{r\sqrt{n}(c+1)}$. Note that the number of variables to the system of equations is r^3 which is roughly like $8n$ and the number of equations is $\sqrt{n}\sqrt{n} = n$. Therefore, the above system has infinitely many solutions in integers. Also, observe that $n/(r^3 - n) < 1$.

By Siegel's Lemma 2.1, there exists a non-zero integer solution, say, (a_{uvw}) to the above system together with

$$|a_{uvw}| \leq (2r^3(30)^{r\sqrt{n}(c+1)})^{n/(r^3-n)} \leq C_1^{r\sqrt{n}},$$

for some positive constant $C_1 = 2(30)^{c+1}$, as $e^{3\log r} < 2^{r\sqrt{n}}$.

Note that not all a_{uvw} is 0. In this way, we construct the function $F(z)$ in (3.1)

By (3.2), we see that the entire function $F(z) = 0$ for all $z = ac + b$ with integers $1 \leq a, b \leq \sqrt{n}$. Since 2, 3 and 5 are distinct primes, log 2, log 3 and log 5 are $\mathbb{Q}$-linearly independent numbers. Therefore, by (2.2.3), the complex functions $e^{(\log 2)z}, e^{(\log 3)z}$ and $e^{(\log 5)z}$ are algebraically independent as functions over $\mathbb{C}$. That is, the complex functions 2^z, 3^z and 5^z are algebraically independent as functions over $\mathbb{C}$. Since not all a_{uvw} are zero, we conclude that $F(z)$ is a non-zero entire function.

It may happen that $F(cx + y) = 0$ for all integers $x, y \in \mathbb{Z}$. Since c is an irrational number, the set $S := \{cx + y : x, y \in \mathbb{Z}\}$ is dense in $\mathbb{R}$.

Also, note that since c is not an integer and 2^c is an integer, we see that c cannot be rational number. Therefore, c is an irrational number.

It may happen that $F(cx + y) = 0$ for all integers $x, y \in \mathbb{Z}$. Since c is an irrational number, by (2.2.4), the set $S := \{cx + y : x, y \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Since F is an entire function in $\mathbb{C}$ and $F(\alpha) = 0$ for all $\alpha \in S$, by continuity, we conclude that $F(\beta) = 0$ for every $\beta \in \mathbb{R}$. Hence, by the identity theorem (2.2.1), we see that F is identically zero on $\mathbb{C}$, which is a contradiction.

Hence, there exists a least positive integer s such that

$$F(ac + b) = 0 \text{ for all } 1 \leq a, b \leq s \text{ and } F(a'c + b') \neq 0 \quad (3.3)$$

where either $a' = s+1$ and $1 \leq b' \leq s$ or $b' = s+1$ and $1 \leq a' \leq s$.

Let $z_0 = a'c + b'$ with $a' = s + 1$ and $1 \leq b' \leq s$ (and the other case is similar). Then note that $F(z_0)$ is a non-zero integer and let $m_0 = F(z_0)$. Then m_0 is the required integer satisfying

$$|m_0| \geq 1. \quad (3.4)$$

Now, we need to get the upper bound for $|m_0|$. This can be done using complex analysis as follows.

Since F is an entire function, we estimate $|F(z)|$ on $|z| = R$ for any real number $R > 0$. For any $z \in \mathbb{C}$ satisfying $|z| = R$, we have

$$\begin{aligned}
|F(z)| &\leq \sum_{u=1}^{r}\sum_{v=1}^{r}\sum_{w=1}^{r} |a_{uvw}||2^{uz}||3^{vz}||5^{wz}| \\
&\leq \max|a_{uvw}| \sum_{u=1}^{r}\sum_{v=1}^{r}\sum_{w=1}^{r} 2^{r|z|}3^{r|z|}5^{r|z|} \\
&\leq C_1^{r\sqrt{n}} r^3 (30)^{r|z|} \leq r^3 C_1^{r(\sqrt{n}+R)}.
\end{aligned}$$

Thus, for any real number $R > 0$, we have

$$|F(z)| \leq r^3 C_1^{r(\sqrt{n}+R)} \text{ for all } |z| = R \text{ where } C_1 = 2(30)^{c+1}. \quad (3.5)$$

Since $m_0 = F(a'c + b')$ and $|a'c + b'| \leq 2(c + 1)s$, we choose $R = s^{1+\frac{1}{8}}$ and $k = 2(c + 1)s$. Then, we see that $|z_0| = |a'c + b'| \leq$

$k < R$ for all large enough n's. Also note that in the disk $|z| \leq k$, the analytic function F has at least s^2 number of zeros, namely, $F(ac + b) = 0$ for all integers $1 \leq a, b \leq s$. Therefore, by the Schwarz's Lemma 2.2.2, we get

$$|m_0| = |F(z_0)| \leq \left(\frac{6(c+1)s}{s^{1+\frac{1}{8}}}\right)^{s^2} \|F\|_R \leq \left(\frac{6(c+1)}{s^{1/8}}\right)^{s^2} \|F\|_R. \quad (3.6)$$

By (3.5), we get,

$$\|F\|_R \leq r^3 C_1^{r(\sqrt{n}+R)}. \quad (3.7)$$

Therefore, by (3.4), (3.6) and (3.7), we get,

$$1 \leq |m_0| \leq \left(\frac{6(c+1)}{s^{1/8}}\right)^{s^2} r^3 C_1^{r(\sqrt{n}+R)}.$$

By taking log both sides, we get,

$$0 \leq s^2 \log(6(c+1)) - \frac{s^2}{8} \log s + 3 \log r + (r(\sqrt{n}+R)) \log C_1. \quad (3.8)$$

Since $r = 2n^{1/3}$, $s \geq \sqrt{n}$ and $R = s^{1+\frac{1}{8}}$, we see that $\sqrt{n} \leq s \leq R$, $r \leq 2n^{1/3} \leq 2s^{2/3}$ and

$$r(\sqrt{n} + R) \leq 2rR \leq 4s^{\frac{2}{3}} s^{1+\frac{1}{8}} \leq 4s^{1+\frac{19}{24}}.$$

Therefore, by (3.8), we get,

$$0 \leq s^2 \log(6(c+1)) - \frac{s^2}{8} \log s + 3 \log r + 4s^{1+\frac{19}{24}} \log C_1.$$

That is, we get,

$$\frac{s^2}{8} \log s \leq s^2 \log(6(c+1)) + 4s^{1+\frac{19}{24}} \log 2 + 4s^{1+\frac{19}{24}} (c+1) \log 30.$$

Finally, we arrive at

$$s^{5/24} \left(\frac{1}{8} \log s - \log(6(c+1))\right) \leq 4 \log 2 + 4(c+1) \log 30.$$

Note that right hand side of the above inequality is bounded and left hand side can be made as large as possible by choosing the parameter n very large. Since $s \geq \sqrt{n}$, the above inequality is not possible. This proves Theorem 1.

One notes that if we take 2 and 3, instead of three primes, and we apply the above method, then the inequality (3.8) fails to produce $|m_0| < 1$.

Acknowledgement

The second author is thankful to Ms. Bidisha Roy for going through the manuscript carefully and for useful suggestions. We would like to thank Prof. B Sury for his constant support and for modifying the draft manuscript to a more readable form.

Suggested Reading

[1] K Ramachandra, Contributions to the Theory of Transcendental Numbers. I, II, *Acta Arith*., 14, pp.65–72, pp.73–88, 1967–68.
[2] S Lang, *Introduction to Transcendental Numbers*, Addison–Wesley, 1966.
[3] L Alaoglu and P Erdős, On Highly Composite and Similar Numbers, *Trans. Amer. Math. Soc*., 56, pp.448–469, 1944.
[4] C L Siegel, Über Einige Anwendungen Diophantischer Approximationen, *Abh. Preuss. Akad. Wiss. Phys. Math*, pp.41–69. Reprinted in Gesammelte Abhandlungen, Volume 1, 1929.
[5] S Lang, *Complex Analysis*, GTM 103, Fourth Edition, Springer–Verlag, 1999.
[6] K Senthil Kumar, R Thangadurai and M Waldschmidt, Liouville Numbers and Schanuel's conjecture, *Arch. Math*., 102, pp.59–70, 2014.
[7] M Ram Murty, *Problems in Analytic Number Theory*, GTM 206, Readings in Mathematics, Springer, New York, 2008.
[8] A Baker, *Transcendental Number Theory*, Cambridge University Press, 1975.

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