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On zero-sum subsequences in a finite abelian p-group of length not exceeding a given number

Bidisha Roy *, R. Thangadurai

Harish-Chandra Research Institute, HBNI, Chhatnag Road, Jhunsi, Allahabad 211019, India

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ABSTRACT

Let G be a finite abelian group. For any integer $a \geq 1$, we define the constant $s_{\leq a}(G)$ as the least positive integer t such that any sequence S over G of length at least t has a zero-sum subsequence of length $\leq a$ in it. In this article, we compute this constant for many classes of abelian p-groups. In particular, it proves a conjecture of Schmid and Zhuang [20].

1. Introduction

Let G be a finite abelian additive group with exponent $\exp(G)$. A sequence S over G is written as

$$S = \prod_{i=1}^{|S|} g_i = \prod_{g \in G} g^{v_g(S)} \text{ with } v_g(S) \in \mathbb{Z}_{\geq 0}$$

* Corresponding author.
E-mail addresses: bidisharoy@hri.res.in (B. Roy), thanga@hri.res.in (R. Thangadurai).

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where $v_g(S)$ is called the *multiplicity* of g in S and $|S|$ denotes the length of the sequence S. By the definition of multiplicity, we see that

$$|S| = \sum_{g \in G} v_g(S) \in \mathbb{Z}_{\geq 0}.$$

The sum of all the terms of the sequence S is given by

$$\sigma(S) = \sum_{g \in G} v_g(S) g \in G.$$

A sequence S over G is called a *zero-sum sequence* if $\sigma(S) = 0$. For any integer $k \in \mathbb{Z}_{>0}$ and for a sequence S over G, we define

$$N^k(S) = \left| \left\{ I \subset [1, |S|] \ : \ \sum_{i \in I} g_i = 0, |I| = k \right\} \right|,$$

which denotes the number of zero-sum subsequences, counted with multiplicities, of S of length k.

For a given positive integer $k \geq 1$, we define a constant $s_{\leq k}(G)$ which is the least positive integer t such that given any sequence S over G of length $|S| \geq t$ satisfies $N^m(S) \geq 1$ for some integer $1 \leq m \leq k$. The well-known Davenport constant, $D(G)$, is defined as the least positive integer t such that any given sequence S over G of length $\geq t$ satisfies $N^k(S) \geq 1$ for some integer $k \geq 1$. The other well-known constant $\eta(G)$ is nothing but $\eta(G) = s_{\leq \exp(G)}(G)$.

These constants $D(G)$ and $\eta(G)$ have received a lot of attention (see for instance [1,4,5,7–9,11,13–15,20,21]). When G is a cyclic group, we have $\eta(G) = |G|$ and $D(G) = |G|$. When $G \cong C_p^2$ for a prime p, Olson [18,19] proved in 1969 that $\eta(C_p^2) = 3p - 2$ and for any p-group G, he proved that $D(G) = D^*(G)$ where, for any finite abelian group $G' \cong C_{m_1} \oplus \cdots \oplus C_{m_r}$ with $1 < m_1 \leq m_2 \leq \cdots \leq m_r$ are integers satisfying $m_i | m_{i+1}$, the constant $D^*(G')$ is defined by

$$D^*(G') = 1 + \sum_{i=1}^{r} (m_i - 1).$$

If $G \cong C_m \oplus C_n$ with $m|n$ is an abelian group of rank 2, then it is known that $\eta(G) = 2m + n - 2$ as given in [15] and $D(G) = m + n - 1$.

When G is of rank ≥ 3, nothing much is known. For any odd prime p, it is known that $\eta(C_p^3) \geq 8p - 7$ ([5]) and $\eta(C_p^4) \geq 19p - 18$ ([4]) and their exact values are still unkonwn. Recently, Fan, Gao, Wang and Zhong [7] determined the value $\eta(G)$ for special types of abelian groups of rank 3. Apart from these results, Schmid and Zhuang [20] proved that if G is a finite abelian p-group with $D(G) = 2\exp(G) - 1$, then $\eta(G) = 2D(G) - \exp(G)$. Moreover, they conjectured the following.

Conjecture 1. *([20]) Let G be a finite abelian p-group with $D(G) \leq 2\exp(G) - 1$. Then*

$$\eta(G) = 2D(G) - \exp(G).$$

The constants $s_{\leq k}(G)$ was introduced by Delorme, Ordaz and Quiroz [3]. It is easy to see that if $k \geq D(G)$, then $s_{\leq k}(G) = D(G)$ and if $1 \leq k < \exp(G)$, we see that $s_{\leq k}(G) = \infty$. In general, the problem of determining exact value of $s_{\leq k}(G)$ is quite difficult. In 2010, Freeze and Schmid [10] proved that $s_{\leq 3}(C_2^r) = 2^{r-1} + 1$. In 2017, Wang and Zhao [22] proved that when $G = C_m \oplus C_n$, the constant $s_{\leq D(G)-k}(G)$ is equal to $D(G)+k$ for all integers $k \in [0, m-1]$ and $s_{r-k}(C_2^r) = r+2$ for all $r-k \in \left[\left\lceil \frac{2r+2}{3} \right\rceil, r\right]$.

By the definition of $\eta(G)$, it is clear that $s_{\leq \exp(G)+\ell}(G) \leq \eta(G)$ for all integers $\ell \geq 0$. In this article, we prove that $s_{\leq \exp(G)+\ell}(G) \leq \eta(G) - \ell$ for many classes of finite abelian p-groups and for many integers $\ell \geq 0$. In particular, we get the following results.

- For many classes of finite abelian p-groups G, we get $\eta(G) = 2D(G) - \exp(G)$, which proves Conjecture 1. More recently, S. Luo [17] proved Conjecture 1 using entirely different method.
- When $G \cong C_{p^m} \oplus C_{p^n}$ with $n \geq m + 1$, we get

$$s_{\leq \exp(G)+\ell}(G) = 2D(G) - \exp(G) - \ell$$

for all integers $0 \leq \ell \leq p^m - 1$, which matches with the result of Wang and Zhao [22].

More precisely, we prove the following theorem.

Theorem 1.1. *Let H be a finite abelian p-group with exponent $\exp(H) = p^m$ for some integer $m \geq 1$ and for a prime number $p > 2r(H)$ where $r(H)$ is the rank of H. Suppose the Davenport constant $D(H)$ satisfies $D(H) - 1 = kp^m + t$ for some integers $k \geq 1$ and $0 \leq t \leq p^m - 1$. Let $G = C_{p^n} \oplus H$ be a finite abelian p-group for some integer n satisfying $p^n \geq 2(D(H)-1)$. Let ℓ be any integer satisfying $\ell = ap^m + t'$ for some integer a satisfying $0 \leq a \leq k-1$ and for some integer t' satisfying $0 \leq t' \leq t$. Then, we have*

$$s_{\leq \exp(G)+\ell}(G) \leq \exp(G) + 2(D(H) - 1) - \ell = 2D(G) - \exp(G) - \ell.$$

In particular, we get $\eta(G) = 2D(G) - \exp(G)$; when $H \cong C_{p^m}$ and $n \geq m + 1$, for all integers $0 \leq \ell \leq p^m - 1$, we get

$$s_{\leq \exp(G)+\ell}(G) = 2D(G) - \exp(G) - \ell.$$

Earlier, in 2016, Gao, Han and Zhang [12] proved Conjecture 1 for the abelian p-groups G satisfying $p > 2r(H)$ and $\left\lceil \frac{2D(H)}{\exp(H)} \right\rceil$ is either even or at most 3. Recently, Chintamani, Paul and Thangadurai [2] considered similar problem for the complementary case that of [12] and obtained an upper bound. By refining the method employed in [12], we shall prove Theorem 1.1.

2. Preliminaries

We shall start with the following useful lemmas.

Lemma 2.1. *([12]) Let G be a finite abelian p-group and let m be a positive integer. If S is a sequence over G of length $|S| \geq D(G) + p^m - 1$, then we have*

$$1 + \sum_{j=1}^{\left\lfloor \frac{|S|}{p^m} \right\rfloor} (-1)^j N^{jp^m}(S) \equiv 0 \pmod{p}.$$

Lemma 2.2. *([6]) Let H be a finite abelian p-group with $D(H) \leq p^n - 1$ and let $G = C_{p^n} \oplus H$. Then, $D(G) = p^n + D(H) - 1 = \exp(G) + D(H) - 1$.*

Lemma 2.3. *([20]) Let G be any finite abelian p-group with exponent $\exp(G)$ such that $D(G) \leq 2\exp(G) - 1$. Then $\eta(G) \geq 2D(G) - \exp(G)$.*

Throughout this section, now on, we take H to be a finite abelian p-group of rank $r(H)$ and exponent $\exp(H) = p^m$ for some positive integer m. Also, we write $D(H) - 1 = kp^m + t$ for some positive integer k and a non-negative integer t satisfying $0 \leq t \leq p^m - 1$. Choose any integer n such that $p^n \geq 2(D(H) - 1)$ and let $G = C_{p^n} \oplus H$. Let ℓ be any integer satisfying $\ell = ap^m + t'$ for some integer a with $0 \leq a \leq k - 1$ and for some integer t' with $0 \leq t' \leq t$.

We need the following lemma which was proved in ([12]) for the case when $\ell = 0$. We prove for all integers ℓ satisfying as above.

Lemma 2.4. *Let $v = (k+1)p^m - D(H) = p^m - t - 1$. Let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1) - \ell$ such that $N^b(S) = 0$ for all integers b with $1 \leq b \leq p^n + \ell$. Then for any integers $i \in [0, k - a - 1]$, $h \in [0, v + \ell]$ or $i = k - a$ and $h = v + \ell$ and for any subsequence T of S of length $|T| = |S| - ip^m$, we have*

$$1 + \sum_{u=0}^{h} \binom{h}{u} \sum_{j=a+1}^{k} (-1)^{j-1} N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p}. \tag{1}$$

Proof. First, we claim the following.

Claim. $N^i(S) = 0$ for all $i \in [1, p^n + \ell] \cup [p^n + D(H), |S|]$.

Since S has no zero-sum subsequence of length $\leq p^n + \ell$, by the hypothesis, we assume that $N^i(S) \neq 0$ for some integer $i \in [p^n + D(H), |S|]$. Let W be a subsequence of S of length $|W| = i \geq p^n + D(H)$. Since $D(G) = p^n + D(H) - 1$, there exist two disjoint zero-sum subsequences W_1 and W_2 such that $|W_1| \leq |W_2|$ and $W = W_1W_2$. Since $N^j(S) = 0$ for any $j \in [1, p^n + \ell]$, it is clear that $|W_x| \geq p^n + \ell + 1$ for all integers $x = 1, 2$. Therefore, $|S| \geq |W| = |W_1| + |W_2| \geq 2p^n + 2\ell + 2$, which is a contradiction to the assumption that $|S| \leq p^n + 2(D(H) - 1) \leq 2p^n$. Therefore, we get the claim.

In order to get those congruences, we need to apply Lemma 2.1 suitably. In order to apply Lemma 2.1, we shall consider the finite abelian group $G' = G \oplus C_{p^m}$ and consider the map $f : G \to G'$ given by $f(g) = g + e$ where e is a generator of the cyclic group C_{p^m}. Under this map, we consider the image of the given sequence $f(S)$.

Let i be a fixed integer with $0 \leq i \leq k - a - 1$. Let T be a subsequence of S of length $|T| = |S| - ip^m = p^n + 2(D(H) - 1) - \ell - ip^m$. Let h be a fixed integer with $0 \leq h \leq v + \ell$ and consider the sequence $T0^h$. Then,

$$\begin{aligned}|T0^h| &= |T| + h = p^n + D(H) - 1 + D(H) - 1 + h - \ell - ip^m \\ &= D(G) + kp^m + t + h - ap^m - t' - ip^m \\ &= D(G) + (k - a - i)p^m + t - t' + h \\ &\geq D(G) + p^m\end{aligned}$$

holds true for all integers $i \in [0, k - a - 1]$ and for all integers $h \in [0, v + \ell]$ as $t' \leq t$. Also, when $i = k - a$, we take $h = v + \ell$ so that we get

$$|T0^{v+\ell}| = D(G) + t - \ell + v + \ell = D(G) + t + p^m - t - 1 = D(G) + p^m - 1.$$

Now, we apply Lemma 2.1 to the sequence $f(T0^h)$ to get

$$1 + \sum_{j=1}^{z} (-1)^j N^{jp^m}(f(T0^h)) \equiv 0 \pmod{p} \tag{2}$$

where $z = \left\lfloor \frac{|T0^h|}{p^m} \right\rfloor$, for all integers $i \in [0, k - a - 1]$ and $h \in [0, v + \ell]$ and when $i = k - a$, take $h = v + \ell$. Note that for each integer $j = 1, 2, \ldots, z$, we have

$$N^{jp^m}(f(T0^h)) = \sum_{u=0}^{h} \binom{h}{u} N^{jp^m - u}(T).$$

Therefore, for all integers $i \in [0, k - a - 1]$ and $h \in [0, v + \ell]$ or when $i = k - a$, we take $h = v + \ell$, we get,

$$1 + \sum_{u=0}^{h} \binom{h}{u} \sum_{j=1}^{z} (-1)^{j-1} N^{jp^m - u}(T) \equiv 0 \pmod{p}.$$

Since, by claim, we know that $N^b(T) = 0$ for all $b \in [1, p^n + \ell] \cup [p^n + D(H), |T|]$, and $p^n + D(H) = p^n + (k+1)p^m - v$, we get

$$1 + \sum_{u=0}^{h} \binom{h}{u} \sum_{j=a+1}^{k} (-1)^{j-1} N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p}$$

is true for all integers $i \in [0, k-a-1]$ and $h \in [0, v+\ell]$ and when $i = k-a$, take $h = v + \ell$. From this, we get the required congruences. □

Now, we shall prove the following refinement of Lemma 3.1 (3.3) in [12].

Lemma 2.5. *Let $v = (k+1)p^m - D(H) = p^m - t - 1$. Let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1) - \ell$ for some integer ℓ satisfying $\ell = ap^m + t'$ for some integer a with $0 \le a \le k-1$ and for some integer t' with $0 \le t' \le t$ such that $N^b(S) = 0$ for all integers b with $1 \le b \le p^n + \ell$. For any integers i and h satisfying $0 \le i \le k-a-1$ and $0 \le h \le v + \ell$, we have*

$$\binom{|S|}{ip^m} + \sum_{j=a+1}^{k} (-1)^{j-1} \sum_{u=0}^{h} \binom{h}{u} \binom{|S| - p^n - jp^m + u}{ip^m} N^{p^n + jp^m - u}(S) \equiv 0 \pmod{p}, \tag{3}$$

and

$$\binom{|S|}{(k-a)p^m} + \sum_{u=0}^{v+\ell} \binom{v+\ell}{u} \sum_{j=a+1}^{k} (-1)^{j-1} \binom{|S| - p^n - jp^m + u}{(k-a)p^m} N^{p^n + jp^m - u}(S) \equiv 0 \pmod{p}. \tag{4}$$

Proof. In order to get (3), we take a subsequence T of S such that $|T| = |S| - ip^m$ for a given integer i with $0 \le i \le k-a-1$. Then for any integer $h \in [0, v+\ell]$, by (1), we get

$$1 + \sum_{u=0}^{h} \binom{h}{u} \sum_{j=a+1}^{k} (-1)^{j-1} N^{p^n + jp^m - u}(T) \equiv 0 \pmod{p}.$$

Now we sum over all the subsequences T with $|T| = |S| - ip^m$ and we get

$$\sum_{T, |T| = |S| - ip^m} \left(1 + \sum_{u=0}^{h} \binom{h}{u} \sum_{j=a+1}^{k} (-1)^{j-1} N^{p^n + jp^m - u}(T) \right) \equiv 0 \pmod{p}. \tag{5}$$

Since each subsequence W of S with $|W| \le |S| - ip^m$ can be extended to a subsequence T of length $|T| = |S| - ip^m$ in

$$\binom{|S|-|W|}{|T|-|W|} = \binom{|S|-|W|}{|S|-|T|} = \binom{|S|-|W|}{ip^m}$$

ways, by starting with 0 length subsequence W of S, we see that the number of ways to get subsequences T of S with $|T| = |S| - ip^m$ is $\binom{|S|}{ip^m}$. Then, using this and expanding the sum in (5), we arrive at (3). To get (4), we put $i = k - a$ and $h = v + \ell$ in (1) and apply the same procedure. This proves the lemma. □

Corollary 2.1. *Let S be a sequence over G as defined in Lemma 2.5. For any integer i with $0 \leq i \leq k-a-1$ and for every integer h with $1 \leq h \leq v+\ell$, we have*

$$\binom{|S|}{ip^m} + \sum_{j=a+1}^{k} (-1)^{j-1} \binom{|S|-p^n-jp^m}{ip^m} N^{p^n+jp^m}(S) \equiv 0 \pmod{p} \tag{6}$$

and

$$\sum_{j=a+1}^{k} (-1)^{j-1} \binom{|S|-p^n-jp^m+h}{ip^m} N^{p^n+jp^m-h}(S) \equiv 0 \pmod{p}. \tag{7}$$

Proof. To prove (6), we put $h = 0$ in (3) (Lemma 2.5) and we get the congruence.

We shall prove (7) by induction on h. When $h = 1$, by (3) (Lemma 2.5), we get,

$$\binom{|S|}{ip^m} + \sum_{j=a+1}^{k} (-1)^{j-1} \left[\binom{1}{0} \binom{|S|-p^n-jp^m}{ip^m} N^{p^n+jp^m}(S) \right.$$
$$\left. + \binom{1}{1} \binom{|S|-p^n-jp^m+1}{ip^m} N^{p^n+jp^m-1}(S) \right] \equiv 0 \pmod{p}.$$

Therefore, by (6), we get (7) with $h = 1$.

Suppose we assume (7) is true for all integers $b < h$ and we shall prove for h. We shall rewrite (3) with h as follows.

$$\binom{|S|}{ip^m} + \sum_{j=a+1}^{k} (-1)^{j-1} \sum_{b=0}^{h} \binom{h}{b} \binom{|S|-p^n-jp^m+b}{ip^m} N^{p^n+jp^m-b}(S) \equiv 0 \pmod{p}$$

$$\implies \binom{|S|}{ip^m} + \sum_{b=0}^{h-1} \binom{h}{b} \sum_{j=a+1}^{k} (-1)^{j-1} \binom{|S|-p^n-jp^m+b}{ip^m} N^{p^n+jp^m-b}(S)$$

$$+ \sum_{j=a+1}^{k} (-1)^{j-1} \binom{|S|-p^n-jp^m+h}{ip^m} N^{p^n+jp^m-h}(S) \equiv 0 \pmod{p}$$

By applying the induction hypothesis, we get,

$$\sum_{j=a+1}^{k}(-1)^{j-1}\binom{|S|-p^n-jp^m+h}{ip^m}N^{p^n+jp^m-h}(S)\equiv 0 \pmod p$$

as required. □

The following theorems are very crucial for proving our main result. We record them as follows.

Theorem 2.1. *([16]) Let p be a prime number. Let a and b be positive integers with $a = a_np^n+a_{n-1}p^{n-1}+\cdots+a_0$ with $a_i \in \{0,1,\dots,p-1\}$ and $b = b_np^n+b_{n-1}p^{n-1}+\cdots+b_0$ with $b_i \in \{0,1,\dots,p-1\}$. Then*

$$\binom{a}{b}\equiv\binom{a_n}{b_n}\binom{a_{n-1}}{b_{n-1}}\cdots\binom{a_0}{b_0} \pmod p,$$

where $\binom{a_i}{b_i}=0$, if $a_i<b_i$ and $\binom{0}{0}=1$.

Theorem 2.2. *([12]) Let n and k be positive integers with $1 \le 2k \le n$. Let A be the following $(k+1)\times(k+1)$ matrix with positive integers*

$$A=\begin{pmatrix} 1 & 1 & \cdots & 1\\ \binom{n}{1} & \binom{n-1}{1} & \cdots & \binom{n-k}{1}\\ \binom{n}{2} & \binom{n-1}{2} & \cdots & \binom{n-k}{2}\\ \cdots & & \cdots & \\ \binom{n}{k} & \binom{n-1}{k} & \cdots & \binom{n-k}{k}\end{pmatrix}.$$

Then, the determinant of A is given by

$$det(A)=\left(\prod_{t=1}^{k}t!\right)^{-1}\prod_{1\le i<j\le k}(i-j).$$

The following is the crucial observation for the proof of Theorem 1.1.

Theorem 2.3. *Let S be a sequence over G which is defined as in Lemma 2.5 and let p be a prime number satisfying $p > 2r(H)$. Then for every integer $j \in [a+1,k]$ and for every integer $h \in [1, v+\ell]$, we get,*

$$N^{p^n+jp^m-h}(S)\equiv 0 \pmod p.$$

Proof. Since $p^n \geq 2(D(H)-1) = 2(kp^m+t)$ and $p > 2r(H)$, we see that $2k+1 < p$. Let h be a fixed integer such that $1 \leq h \leq v+\ell$. For any integer $j = a+1, a+2, \ldots, k$, we see that

$$|S| - p^n - jp^m + h = p^n + 2(kp^m+t) - p^n - jp^m + h - \ell = (2k-j)p^m + 2t + h - \ell.$$

Note that

$$2t + h - \ell \leq 2t + v + \ell - \ell = 2t + p^m - t - 1 = t + p^m - 1 \leq p^m - 1 + p^m - 1 = 2p^m - 2,$$

as $t \leq p^m - 1$. Hence, for each integer $j = a+1, a+2, \ldots, k$, we see that

$$|S| - p^n - jp^m + h = (2k - j + c)p^m + f$$

where $c = 0$ or 1 depending on values t and h and for some integer $0 \leq f < p^m$. Therefore, by Theorem 2.1, we get

$$\binom{|S| - p^n - jp^m + h}{ip^m} = \binom{(2k-j)p^m + 2t + h}{ip^m} \equiv \binom{2k-j+c}{i} \pmod{p} \tag{8}$$

for all integers $j = a+1, a+2, \ldots, k$ and $i = 0, 1, \ldots, k-a-1$ where $c = 0$ or 1.

Let h be a fixed integer with $1 \leq h \leq v + \ell$ and let

$$X_j = (-1)^{j-1} N^{p^n + jp^m - h}(S)$$

for every integer $j = a+1, a+2, \ldots, k$. Then by the congruence (7) in Corollary 2.1, we get a system of $k-a$ linear equations in $k-a$ variables over $\mathbb{F}_p$ as follows.

$$\begin{aligned}
&X_{a+1} + X_{a+2} + \cdots + X_k = 0;\\
&\binom{|S| - p^n - p^m + h}{p^m} X_{a+1} + \binom{|S| - p^n - 2p^m + h}{p^m} X_{a+2} + \cdots\\
&\quad + \binom{|S| - p^n - kp^m + h}{p^m} X_k = 0;\\
&\cdots \qquad \cdots \qquad \cdots\\
&\binom{|S| - p^n - p^m + h}{(k-a-1)p^m} X_{a+1} + \binom{|S| - p^n - 2p^m + h}{(k-a-1)p^m} X_{a+2} + \cdots\\
&\quad + \binom{|S| - p^n - kp^m + h}{(k-a-1)p^m} X_k = 0;
\end{aligned}$$

By (8), the coefficient matrix of the above system of linear equations over $\mathbb{F}_p$ is

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \binom{2k-a-1+c}{1} & \binom{2k-a-2+c}{1} & \cdots & \binom{2k-k+c}{1} \\ \binom{2k-a-1+c}{2} & \binom{2k-a-2+c}{2} & \cdots & \binom{2k-k+c}{2} \\ \cdots & & \cdots & \\ \binom{2k-a-1+c}{k-a-1} & \binom{2k-a-2+c}{k-a-1} & \cdots & \binom{2k-k+c}{k-a-1} \end{pmatrix}$$

whose determinant, by Theorem 2.2, is non-zero modulo p, by taking $n = 2k - 1 + c$ in Theorem 2.2. Hence the only solution of the above system is $X_{n+1} = \cdots = X_k = 0$ in $\mathbb{F}_p$. This proves the theorem. □

3. Proof of Theorem 1.1

We prove that $s_{\leq p^n+\ell}(G) \leq p^n+2(D(H)-1)-\ell$ for all integers ℓ satisfying $\ell = ap^m+t'$ for some integer a with $0 \leq a \leq k-1$ and for some integer t' with $0 \leq t' \leq t$ where t is an integer satisfying $D(H) - 1 = kp^m + t$ with $0 \leq t \leq p^m - 1$.

Let S be a sequence over G of length $|S| = p^n + 2(D(H) - 1) - \ell$. Suppose that $N^b(S) = 0$ for all integers $1 \leq b \leq p^n + \ell$. Then, by Theorem 2.3, we know that

$$N^{p^n+jp^m-h}(S) \equiv 0 \pmod p$$

for all integers $j \in [a+1, k]$ and integers $h \in [1, v+\ell]$. Therefore, by Lemma 2.5, we get,

$$\binom{|S|}{(k-a)p^m} + \sum_{j=a+1}^{k} (-1)^{j-1} \binom{|S| - p^n - jp^m}{(k-a)p^m} N^{p^n+jp^m}(S) \equiv 0 \pmod p \tag{9}$$

and by Corollary 2.1 (6), we get,

$$\binom{|S|}{ip^m} + \sum_{j=a+1}^{k} (-1)^{j-1} \binom{|S| - p^n - jp^m}{ip^m} N^{p^n+jp^m}(S) \equiv 0 \pmod p \tag{10}$$

holds true for all integers $i \in [0, k-a-1]$.

Now, we put

$$X_j = (-1)^{j-1} N^{p^n+jp^m}(S)$$

for all $j = a+1, a+2, \ldots, k$ and $X_a = 1$. Then, by (9) and (10), we get a system of $(k-a+1)$ linear equations in $(k-a+1)$ unknowns over $\mathbb{F}_p$ as follows.

$$\binom{|S|}{0} X_a + \binom{|S| - p^n - p^m}{0} X_{a+1} + \cdots + \binom{|S| - p^n - kp^m}{0} X_k \equiv 0 \pmod p;$$

$$\cdots \quad \cdots \quad \cdots$$

$$\binom{|S|}{(k-a-1)p^m}X_a+\binom{|S|-p^n-p^m}{(k-a-1)p^m}X_{a+1}+\cdots+\binom{|S|-p^n-kp^m}{(k-a-1)p^m}X_k \equiv 0 \pmod p;$$
$$\binom{|S|}{(k-a)p^m}X_a+\binom{|S|-p^n-p^m}{(k-a)p^m}X_{a+1}+\cdots+\binom{|S|-p^n-kp^m}{(k-a)p^m}X_k \equiv 0 \pmod p.$$

Now, we need to compute the determinant of the coefficient matrix of the above system. We shall prove that this determinant is non-zero modulo p, which in turn implies that the only solution of the above system is $X_a = \cdots = X_k = 0$ in $\mathbb{F}_p$. This is a contradiction to $X_a \not\equiv 0 \pmod p$, which proves the theorem. Hence, we need to compute the coefficients modulo p and its determinant. Since the calculation is the same as in the proof of Theorem 2.3, we omit the details here. This proves the upper bound for $s_{\leq p^n+\ell}(G)$.

Note that when $\ell = 0$, by Lemma 2.2, Lemma 2.3 and by the above upper bound, we get

$$s_{\leq \exp(G)}(G) = s_{\leq p^n}(G) = \eta(G) = p^n + 2(D(H) - 1).$$

Now, we shall assume that $G \cong C_{p^m} \oplus C_{p^n}$ with $n \geq m + 1$. Then $H = C_{p^m}$ and $D(H) - 1 = p^m - 1$. Hence $t = p^m - 1$ and $0 \leq \ell \leq t = p^m - 1$. In order to prove the lower bound for $s_{\leq \exp(G)+\ell}(C_{p^m} \oplus C_{p^n})$, we consider the following sequence

$$S = (0, e)^{p^n-1}(f, 0)^{p^m-1}(f, e)^{p^m-1-\ell}$$

over $G \cong C_{p^m} \oplus C_{p^n}$ of length $p^n + 2(p^m - 1) - \ell = \exp(G) + 2(D(H) - 1) - \ell$, where e is a generator of C_{p^n} and f is a generator of C_{p^m}. If T is a zero-sum subsequence of S of length $\leq p^n + \ell$, then

$$T = (0, e)^a(f, 0)^b(f, e)^c$$

for some non-negative integers a, b and c. Since $p^n \geq pp^m$ with $p \geq 5$ and T is a zero-sum sequence, we see that $a + c = p^n$ and $b + c = p^m$. Therefore, $a + 2c + b = p^n + p^m$. Since $|T| = a + b + c = p^n + z$ where $z \leq \ell$, then we get $c = p^m - z \geq p^m - \ell$, which is a contradiction to the fact that $c \leq p^m - 1 - \ell$. Therefore, $N^b(S) = 0$ for all integers $0 \leq b \leq p^n + \ell$. This proves the lower bound. □

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