Received: 12 June 2021 | Revised: 13 May 2022 | Accepted: 24 July 2022

DOI: 10.1112/mtk.12161

RESEARCH ARTICLE

Mathematika

On simultaneous approximation of algebraic numbers

Veekesh Kumar[1] | **R. Thangadurai**[2]

[1]Institute of Mathematical Sciences, A CI of Homi Bhabha National Institute, C.I.T Campus, Taramani, Chennai, India

[2]Harish-Chandra Research Institute, A CI of Homi Bhabha National Institute, Jhunsi, Prayagraj (Allahabad) 211019, India

Correspondence
Veekesh Kumar, Institute of Mathematical Sciences, A CI of Homi Bhabha National Institute, C.I.T Campus, Taramani, Chennai 600 113, India.
Email: veekeshiitg@gmail.com

Present address
Veekesh Kumar, National Institute of Science Education and Research Bhubaneswar, Khurda, Odisha 752050, India

Abstract

Let $\Gamma \subset \overline{\mathbb{Q}}^{\times}$ be a finitely generated multiplicative group of algebraic numbers. Let $\alpha_1, \dots, \alpha_r \in \overline{\mathbb{Q}}^{\times}$ be algebraic numbers which are $\mathbb{Q}$-linearly independent and let $\epsilon > 0$ be a given real number. One of the main results that we prove in this article is as follows: There exist only finitely many tuples $(u, q, p_1, \dots, p_r) \in \Gamma \times \mathbb{Z}^{r+1}$ with $d = [\mathbb{Q}(u) : \mathbb{Q}]$ for some integer $d \geqslant 1$ satisfying $|\alpha_i qu| > 1$, $\alpha_i qu$ is not a pseudo-Pisot number for some integer $i \in \{1, \dots, r\}$ and

$$0 < |\alpha_j qu - p_j| < \frac{1}{H^{\epsilon}(u)|q|^{\frac{d}{r}+\varepsilon}}$$

for all integers $j = 1, 2, \dots, r$, where $H(u)$ is the absolute Weil height. In particular, when $r = 1$, this result was proved by Corvaja and Zannier in [Acta Math. **193** (2004), 175–191]. As an application of our result, we also prove a transcendence criterion which generalizes a result of Hančl, Kolouch, Pulcerová, and Štěpnička in [Czech. Math. J. **62** (2012), no. 3, 613–623]. The proofs rely on the clever use of the subspace theorem and the underlying ideas from the work of Corvaja and Zannier.

MSC 2020
11J68 (primary), 11J87 (secondary)

1 INTRODUCTION

Rational approximation is a fascinating and one of the important techniques to prove transcendental results. This area has very rich history and one of the major milestones is a famous result of Roth [7] extending the earlier works of Thue and Siegel (for a proof, see [8]). Then Ridout [6] proved a p-adic version of Roth, a vast generalization is due to Schmidt [9] (Subspace Theorem) and many versions of the subspace theorem are available now, and these versions are applied to many problems in various branches of Number Theory (for instance, see [10]). In 2004, Corvaja and Zannier [3] proved a "Thue-Roth"-type inequality with "moving targets" to solve a problem of Mahler. Recently in 2019, Kulkarni, Mavraki, and Nguyen [5] proved a generalization of the Mahler problem part of Corvaja and Zannier. In this article, we are interested in the simultaneous approximation of algebraic numbers in the same spirit of Corvaja and Zannier. Also, we apply our main theorem to prove a transcendental result. In order to state our results, we shall start with some terminology.

In order to state our results, we start with the following definition. An algebraic number α is said to be a *pseudo-Pisot number*, if $|\alpha| > 1$, α has integral trace, and all its other conjugates have absolute value strictly less than 1.

Theorem 1.1. *Let $\Gamma \subset \overline{\mathbb{Q}}^{\times}$ be a finitely generated multiplicative group of algebraic numbers. Let $\alpha_1, \ldots, \alpha_r \in \overline{\mathbb{Q}}^{\times}$ be $\mathbb{Q}$-linearly independent algebraic numbers. For a given real number $\varepsilon > 0$, let $\mathcal{B}$ be a subset of $\Gamma \times \mathbb{Z}^{r+1}$ which consists of tuples $(u, q, p_1, \ldots, p_r) \in \Gamma \times \mathbb{Z}^{r+1}$ with $d = [\mathbb{Q}(u) : \mathbb{Q}]$ for some integer $d \geqslant 1$ satisfying $|\alpha_i qu| > 1$, $\alpha_i qu$ is not a pseudo-Pisot number for some integer $i \in \{1, 2, \ldots, r\}$ and*

$$0 < |\alpha_j qu - p_j| < \frac{1}{H^{\varepsilon}(u)|q|^{\frac{d}{r}+\varepsilon}} \quad \textit{for all integers } j = 1, 2, \ldots, r, \tag{1.1}$$

where $H(u)$ is the absolute Weil height of an algebraic number. Then $\mathcal{B}$ is a finite set.

When we put $r = 1$ in Theorem 1.1, we recover one of the main results of Corvaja and Zannier in [3]. As an application of Theorem 1.1, we prove the following transcendence criterion.

Corollary 1.1. *Let $\alpha_1, \ldots, \alpha_r$ be $\mathbb{Q}$-linearly independent real numbers. Let α be an algebraic number of degree $d \geqslant 2$ such that one of its conjugates β satisfies $|\beta| > |\alpha| > 1$. For some $\eta > 0$, suppose that there exist infinitely many tuples $(n, q, p_1, \ldots, p_r) \in \mathbb{Z}^2_{>0} \times \mathbb{Z}^r$ satisfying*

$$0 < \left|\alpha_j q\alpha^n - p_j\right| < \frac{1}{H^{\eta}(\alpha^n) q^{\frac{d}{r}+\eta}} \quad \textit{for all integers } j = 1, \ldots, r,$$

where $H(u)$ is the absolute Weil height of an algebraic number. Then at least one of the numbers among $\alpha_1, \ldots, \alpha_r$ is transcendental.

Corollary 1.1 is an extension of a result due to Hančl, Kolouch, Pulcerová, and Štěpnička in [4]. It is important to note that Corvaja and Hančl [2], in 2007, proved the transcendence of infinite product using the new diophantine approximation result proved by Corvaja and Zannier [3] for the first time in the literature. As an example, with the similar spirits in [4], one can conclude the following transcendental result.

Let $\alpha > 1$ be a real algebraic number of degree $d \geqslant 2$ such that a conjugate β of α satisfying $|\beta| > \alpha$ and let $r \geqslant 2$ be an integer. Let $\delta > 0$ and $\epsilon > 0$ be real numbers satisfying $\frac{1+\frac{d}{r}+\delta}{\frac{d}{r}+1} \cdot \frac{\epsilon}{1+\epsilon} > 1$. Let $(a_n)_{n\geqslant 1}$ be a sequence of positive integers and for $i = 1, 2, \dots, r$, let $(b_n^{(i)})_{n\geqslant 1}$ be a sequence of positive integers. Let $B_n^{(i)} = b_n^{(i)}\alpha^{a_n}$ for all $n \geqslant 1$ and for all $i = 1, 2, \dots, r$. Suppose that the sequence $(B_n^{(i)})_{n\geqslant 1}$ is non-decreasing and satisfying the growth conditions

$$\limsup_{n\to\infty} \left(B_n^{(i)}\right)^{\frac{1}{(2+(d/r)+\delta)^n}} = \infty \text{ and } B_n^{(i)} \gg n^{1+\epsilon}$$

for all $i = 1, 2, \dots, r$. If $\alpha_i = \prod_{n=1}^{\infty} \frac{[B_n^{(i)}]}{B_n^{(i)}}$ for all $i = 1, 2, \dots, r$, then either at least one of the numbers among $\alpha_1, \dots, \alpha_r$ is transcendental or they are $\mathbb{Q}$-linearly dependent. Here $[x]$ denotes the integral part of the real number x.

Let K be a number field and let S be a finite set of places on K such that S contains all the archimedean valuations of K. The group of S-units, denoted by $\mathcal{O}_S^{\times}$, is defined as

$$\mathcal{O}_S^{\times} = \{\alpha \in K : |\alpha|_v = 1 \quad \text{for all } v \notin S\}.$$

Now, we shall state the other result of this article as follows.

Theorem 1.2. *Let K be a number field of degree n which is Galois over $\mathbb{Q}$ and let S be a finite set of places on K such that S contains all the archimedean places of K. Let d be a divisor of n and let $\alpha_1, \dots, \alpha_d \in K$ be given algebraic numbers and not all zero. For a given real number $\varepsilon > 0$, let $\mathcal{B}$ be a subset of $\mathcal{O}_S^{\times} \times \mathbb{Z}^2$ which consists of triples $(u_1, q, p) \in \mathcal{O}_S^{\times} \times \mathbb{Z}^2$ with $d = [\mathbb{Q}(u_1) : \mathbb{Q}]$ for some integer $d \geqslant 1$ satisfying $|\alpha_i q u_i| > 1$, $\alpha_i q u_i$ is not a pseudo-Pisot number for some integer $i \in \{1, 2, \dots, d\}$ and*

$$0 < |\alpha_1 q u_1 + \alpha_2 q u_2 + \cdots + \alpha_d q u_d - p| \leqslant \frac{1}{H^{\varepsilon}(u_1)|q|^{d+\varepsilon}},$$

where $u_1, u_2, \dots, u_d$ are all Galois conjugates. Then $\mathcal{B}$ is a finite set.

2 | PRELIMINARIES

Let $K \subset \mathbb{C}$ be a number field which is Galois over $\mathbb{Q}$ with Galois group $\mathrm{Gal}(K/\mathbb{Q})$. Let M_K be the set of all inequivalent places of K and M_∞ be the set of all archimedean places of K. For each place $v \in M_K$, we denote $|\cdot|_v$ the absolute value corresponding to v, normalized with respect to K. Indeed if $v \in M_\infty$, then there exists an automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ of K such that for all $x \in K$,

$$|x|_v = |\sigma(x)|^{d(\sigma)/[K:\mathbb{Q}]}, \tag{2.1}$$

where $d(\sigma) = 1$ if $\sigma(K) = K \subset \mathbb{R}$ and $d(\sigma) = 2$ otherwise. Note that $d(\sigma)$ is constant since $K/\mathbb{Q}$ is Galois. Non-archimedean absolute values are normalized accordingly so that the product formula $\prod_{\omega\in M_K} |x|_\omega = 1$ holds for any $x \in K^{\times}$.

The absolute Weil height $H(x)$ is defined as

$$H(x) := \prod_{\omega \in M_K} \max\{1, |x|_\omega\} \text{ for all } x \in K.$$

For a vector $\mathbf{x} = (x_1, \dots, x_n) \in K^n$ and for a place $\omega \in M_K$, the ω-norm for $\mathbf{x}$, denoted by $||\mathbf{x}||_\omega$, is defined by

$$\|\mathbf{x}\|_\omega := \max\{|x_1|_\omega, \dots, |x_n|_\omega\}$$

and the projective height, $H(\mathbf{x})$, is defined by

$$H(\mathbf{x}) = \prod_{\omega \in M_K} \|\mathbf{x}\|_\omega.$$

Now we are ready to state a more general version of the Schmidt Subspace Theorem, which was formulated by Evertse and Schlickewei. For a reference, see [1, Chapter 7], [9, Chapter V, Theorem $1D'$], and [10, Page 16, Theorem II.2].

Theorem 2.1 (Subspace theorem). *Let K be a number field and $m \geqslant 2$ be an integer. Let S be a finite set of places on K which contains all the archimedean places of K. For each $v \in S$, let $L_{1,v}, \dots, L_{m,v}$ be linearly independent linear forms in the variables $X_1, \dots, X_m$ with coefficients in K. For any $\varepsilon > 0$, the set of solutions $\mathbf{x} \in K^m$ to the inequality*

$$\prod_{v \in S} \prod_{i=1}^{m} \frac{|L_{i,v}(\mathbf{x})|_v}{\|\mathbf{x}\|_v} \leqslant \frac{1}{H(\mathbf{x})^{m+\varepsilon}}$$

lies in finitely many proper subspaces of K^m.

The following lemma is an application of Theorem 2.1 which can be deduced from results obtained by Evertse. For a proof, we refer to [3, Lemma 1].

Lemma 2.1. *Let K be a number field which is Galois over $\mathbb{Q}$ and S be a finite subset of places which contains all the archimedean places. Let $\sigma_1, \dots, \sigma_n$ be distinct automorphisms of K for some integer $n \geqslant 1$ and let $\lambda_1, \dots, \lambda_n$ be non-zero elements of K. Let $\varepsilon > 0$ be a given real number and $\omega \in S$ be a distinguished place. Let $\mathfrak{E} \subset \mathcal{O}_S^\times$ be a subset which is defined as*

$$\mathfrak{E} := \left\{ u \in \mathcal{O}_S^\times \ : \ |\lambda_1\sigma_1(u) + \cdots + \lambda_n\sigma_n(u)|_\omega < \frac{1}{H^\varepsilon(u)} \max\{|\sigma_1(u)|_\omega, \dots, |\sigma_n(u)|_\omega\} \right\}.$$

If $\mathfrak{E}$ is an infinite subset of $\mathcal{O}_S^\times$, then there exist $a_1, \dots, a_n \in K$, not all zero, such that

$$a_1\sigma_1(v) + \cdots + a_n\sigma_n(v) = 0$$

holds true for infinitely many elements $v \in \mathfrak{E}$.

We also need the following lemma, which is a special case of the S-unit equation theorem proved by Evertse and van der Poorten–Schlickewei. For a proof, we refer to [10, page 18, Theorem II.4].

Lemma 2.2. *Let K, S and $\sigma_1, \dots, \sigma_n$ be as Lemma 2.1. Let $a_1, \dots, a_n$ be non-zero elements of K. Let $\mathfrak{E} \subset \mathcal{O}_S^\times$ be a subset which is defined as*

$$\mathfrak{E} := \left\{ u \in \mathcal{O}_S^\times \;:\; a_1\sigma_1(u) + \cdots + a_n\sigma_n(u) = 0 \right\}.$$

If $\mathfrak{E}$ is an infinite set, then there exist integers $i \neq j$ satisfying $1 \leqslant i < j \leqslant n$, non-zero elements $a, b \in K^\times$ and infinitely many $v \in \mathfrak{E}$ satisfying

$$a\sigma_i(v) + b\sigma_j(v) = 0.$$

We shall start with the following observation.

Lemma 2.3. *Let K be a number field of degree n which is Galois over $\mathbb{Q}$ and $k \subset K$ be a subfield of degree d over $\mathbb{Q}$ for some integer $d \geqslant 2$. Let $\alpha_1, \dots, \alpha_r$ be $\mathbb{Q}$-linearly independent elements of K for some integer $r \geqslant 1$. Let S be a finite set of places on K which contains all the archimedean places and let $\varepsilon > 0$ be a given real number. Let*

$$\mathcal{B} = \left\{ (u, q, p_1, \dots, p_r) \in (\mathcal{O}_S^\times \cap k) \times \mathbb{Z}^{r+1} \;:\; 0 < |\alpha_i q u - p_i| < \frac{1}{H^\varepsilon(u)|q|^{\frac{d}{r}+\varepsilon}} \text{ for all } 1 \leqslant i \leqslant r \right\} \tag{2.2}$$

be a subset of $(\mathcal{O}_S^\times \cap k) \times \mathbb{Z}^{r+1}$. If $\mathcal{B}$ is an infinite set, then $H(u) \to \infty$ as u varies over all the tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}$.

Proof. If possible, $H(u)$ is bounded as u varies over all the tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}$. Then there exists a fixed u, say, u_0 and an infinite subset $\mathcal{A}$ of $\mathcal{B}$ such that if $(u, q, p_1, \dots, p_r) \in \mathcal{A}$, then $u = u_0$ and satisfying

$$0 < |\alpha_i q u_0 - p_i| < \frac{1}{H^\varepsilon(u_0)|q|^{\frac{d}{r}+\varepsilon}}, \quad \text{for all } 1 \leqslant i \leqslant r \tag{2.3}$$

holds true for all tuples $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}$. Since $H(u_0)$ is a constant and if $d/r \geqslant 1$, then, by Roth's theorem, we conclude that $\alpha_1 u_0, \dots, \alpha_r u_0$ are all transcendental which is a contradiction. Hence, we can assume that $d/r < 1$.

Equation (2.3) implies that $\alpha_1 u_0, \dots, \alpha_r u_0$ have simultaneous rational approximation with common denominator q whose exponent is $1 + d/r + \varepsilon = 1 + \delta$ where $\delta \geqslant 1/r$. By the well-known application of the subspace theorem on simultaneous approximation with common denominator, one gets either one of the numbers $\alpha_1 u_0, \dots, \alpha_r u_0$ is transcendental or $1, \alpha_1 u_0, \dots, \alpha_r u_0$ are $\mathbb{Q}$-linearly dependent. Since $\alpha_i u_0$ is algebraic for each integer $i \geqslant 1$, we conclude that $1, \alpha_1 u_0, \dots, \alpha_r u_0$ are $\mathbb{Q}$-linearly dependent. This does not contradict to the fact that $\alpha_1 u_0, \dots, \alpha_r u_0$ are $\mathbb{Q}$-linearly independent. To get a contradiction, we proceed as follows.

Now consider $r+1$ linearly independent linear forms with algebraic coefficients as

$$L_{0,\infty}(x_0, x_1, \dots, x_r) = x_0 \text{ and } L_{i,\infty}(x_0, x_1, \dots, x_r) = \alpha_i u_0 x_0 - x_i$$

for all $i = 1, 2, \dots, r$. We take $K = \mathbb{Q}$ and $S = \{\infty\}$ in Theorem 2.1. Then by (2.3), we conclude that there are infinitely many integer tuples $(q, p_1, p_2, \dots, p_r)$ where $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}$ satisfying Theorem 2.1. Therefore there exist $a_0, a_1, \dots, a_r \in \mathbb{Z}$, not all zero, such that

$$a_0 q + a_1 p_1 + \cdots + a_r p_r = 0 \tag{2.4}$$

holds true for all tuples $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}'$ where $\mathcal{A}'$ is an infinite subset of $\mathcal{A}$. Since not all functions of a_i are 0 in (2.4), we assume that $a_{i_0} \neq 0$ for some integer i_0 satisfying $1 \leqslant i_0 \leqslant r$.

Claim. There exist an infinite subset $\mathcal{A}''$ of $\mathcal{A}'$ and integers $b_1, \dots, b_r$ (not all are zero) such that $b_1 p_1 + \cdots + b_r p_r = 0$ for all the tuples $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}''$

Since $d/r < 1$ and $r, d \geqslant 2$, we see that $\frac{(r-1)d}{r} > 1$. We consider r linearly independent linear forms with algebraic coefficients as

$$L_{i,\infty}(x_1, x_2, \dots, x_r) = \alpha_i u_0 x_{i_0} - x_i \text{ for all } i = 1, 2, \dots, i_0 - 1, i_0 + 1, \dots, r, \text{ and}$$
$$L_{i_0,\infty}(x_1, x_2, \dots, x_r) = x_{i_0}.$$

Now, we take $K = \mathbb{Q}$ and $S = \{\infty\}$ in Theorem 2.1. In order to conclude the assertion of Theorem 2.1, we need to estimate the following quantity:

$$\prod_{i=1}^{r} |L_{i,\infty}(p_1, p_2, \dots, p_{i_0-1}, q, p_{i_0+1}, \dots, p_r)|$$

for all the integer tuples $(p_1, \dots, p_{i_0-1}, q, p_{i_0+1}, \dots, p_r)$ where $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}'$. By (2.3), we get

$$\prod_{i=1}^{r} |L_{i,\infty}(p_1, p_2, \dots, p_{i_0-1}, q, p_{i_0+1}, \dots, p_r)| < \frac{1}{H^{\varepsilon}(u_0)} \frac{|q|}{|q|^{\frac{(r-1)d}{r}+\varepsilon}}$$

holds true for all tuples $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}'$. Since $(r-1)d/r > 1$, we conclude that

$$\prod_{i=1}^{r} |L_{i,\infty}(p_1, p_2, \dots, p_{i_0-1}, q, p_{i_0+1}, \dots, p_r)| < \frac{1}{H^{\varepsilon}(u_0)} \frac{|q|}{|q|^{\frac{(r-1)d}{r}+\varepsilon}} \leqslant \frac{1}{H^{\varepsilon}(u_0)} \frac{1}{|q|^{\varepsilon}}$$

holds true for all tuples $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}'$. Therefore, by Theorem 2.1, we get a non-trivial relation

$$b_{i_0} q + b_1 p_1 + \cdots + b_{i_0-1} p_{i_0-1} + b_{i_0+1} p_{i_0+1} + \cdots + b_r p_r = 0 \quad \text{where} \quad b_i \in \mathbb{Z} \tag{2.5}$$

holds true for all tuples $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}''$ for some infinite subset $\mathcal{A}''$ of $\mathcal{A}'$. Since $\alpha_1, \dots, \alpha_{i_0-1}, \alpha_{i_0+1}, \dots, \alpha_r$ are $\mathbb{Q}$-linearly independent, we conclude that $b_{i_0} \neq 0$. Now, by (2.4) and

(2.5), we can arrive at relation

$$(b_{i_0}a_1 - a_0b_1)p_1 + (b_{i_0}a_2 - a_0b_2)p_2 + \cdots + (b_{i_0}a_{i_0-1} - a_0b_{i_0-1})p_{i_0-1}$$
$$+ b_{i_0}a_{i_0}p_{i_0} + \cdots + (b_{i_0}a_r - a_0b_r)p_r = 0,$$

which holds true for all tuples $(p_1, \dots, p_r)$ where $(u_0, q, p_1, \dots, p_r) \in \mathcal{A}''$. Since a_{i_0} and b_{i_0} are non-zero, we conclude that the above relation is non-trivial. This proves the claim.

Now, dividing by q and letting the tuples $(p_1, \dots, p_r)$ vary over all tuples in $\mathcal{A}''$, we get $\alpha_1 u_0, \dots, \alpha_r u_0$ are $\mathbb{Q}$-linearly dependent, which is a contradiction as $\alpha_1, \alpha_2, \dots, \alpha_r$ are $\mathbb{Q}$-linearly independent. Hence, $H(u) \to \infty$, as desired. □

The following lemmas are key to prove Theorems 1.1 and 1.2.

Lemma 2.4. *Let K be a number field of degree n which is Galois over $\mathbb{Q}$ and $k \subset K$ be a subfield of degree d over $\mathbb{Q}$. Let $\alpha_1, \dots, \alpha_r$ be $\mathbb{Q}$-linearly independent elements of K for some integer $r \geqslant 1$. Let S be a finite set of places on K which contains all the archimedean places and let $\varepsilon > 0$ be a given real number. Let $\mathcal{B}$ be a subset of $(\mathcal{O}_S^\times \cap k) \times \mathbb{Z}^{r+1}$ as defined in (2.2) such that for each tuple $(u, q, p_1, \dots, p_r) \in \mathcal{B}$, there exists an integer $i \in \{1, \dots, r\}$ satisfying $|q\alpha_i u| > 1$ and $q\alpha_i u$ is not a pseudo-Pisot number. If $\mathcal{B}$ is infinite, then there exist a proper subfield $k' \subset k$, a non-zero element $u' \in k$ and an infinite subset $\mathcal{B}' \subset \mathcal{B}$ such that $u/u' \in k'$ for all $(u, q, p_1, p_2, \dots, p_r) \in \mathcal{B}'$.*

Proof. First note that $d \geqslant 2$ as $\mathbb{Q}$ does not admit any proper subfield in it. Let $\mathcal{G} = \mathrm{Gal}(K/\mathbb{Q})$ be the Galois group of K over $\mathbb{Q}$. Since K over k is Galois, we let $\mathcal{H} := \mathrm{Gal}(K/k) \subset \mathcal{G}$ be the subgroup fixing k. Hence, $|\mathcal{G}/\mathcal{H}| = [k : \mathbb{Q}] = d$. Therefore, among the n embeddings of K, there are d embeddings, say, $Id = \sigma_1, \dots, \sigma_d$ which are the complete set of representatives for the left cosets of $\mathcal{H}$ in $\mathcal{G}$ and more precisely, we have

$$\mathcal{G}/\mathcal{H} := \{\mathcal{H}, \sigma_2\mathcal{H}, \dots, \sigma_d\mathcal{H}\}.$$

Each $\rho \in \mathcal{G}$ defines an archimedean valuation on K by the formula

$$|\alpha|_\rho := |\rho^{-1}(\alpha)|^{d(\rho)/[K:\mathbb{Q}]}, \tag{2.6}$$

where $|\cdot|$ denotes the usual absolute value in $\mathbb{C}$. Two elements $\rho_1 \neq \rho_2$ in $\mathcal{G}$ define the same valuation if and only if $\rho_1^{-1} \circ \rho_2$ is the complex conjugation. Then for a fixed i with $1 \leqslant i \leqslant r$, by (2.6), for each $\rho \in \mathrm{Gal}(K/\mathbb{Q})$ and for each tuple $(u, q, p_1, \dots, p_r) \in \mathcal{B}$, we have,

$$|\alpha_i qu - p_i|^{d(\rho)/[K:\mathbb{Q}]} = |\rho(\alpha_i)\rho(qu) - \rho(p_i)|_\rho = |\rho(\alpha_i)q\rho(u) - p_i|_\rho. \tag{2.7}$$

For each $v \in M_\infty$, let ρ_v be an automorphism defining the valuation v, according to (2.6): $|\alpha|_v := |\alpha|_{\rho_v}$; Then the set $\{\rho_v : v \in M_\infty\}$ represents the left cosets of the subgroup generated by the complex conjugation in $\mathcal{G}$. For each $j = 1, 2, \dots, d$, let

$$S_j = \left\{v \in M_\infty \;:\; \rho_v|_k = \sigma_j : k \to \mathbb{C}\right\},$$

and hence $S_1 \cup \dots \cup S_d = M_\infty$. Thus, we have $M_\infty = \{\rho_v : v \in M_\infty\}$ and by (2.7), we get

$$\prod_{v \in M_\infty} |\rho_v(\alpha_i)\rho_v(qu) - p_i|_v = \prod_{j=1}^{d} \prod_{v \in S_j} |\rho_v(\alpha_i)\sigma_j(qu) - p_i|_v. \tag{2.8}$$

By (2.7), we see that

$$\prod_{v\in M_\infty} |\rho_v(\alpha_i)\rho_v(qu)-p_i|_v = \prod_{v\in M_\infty} |\alpha_i qu - p_i|^{d(\rho_v)/[K:\mathbb{Q}]} = |\alpha_i qu - p_i|^{\sum_{v\in M_\infty} d(\rho_v)/[K:\mathbb{Q}]}.$$

Then, from (2.8) and the well-known formula $\sum_{v\in M_\infty} d(\rho_v) = [K : \mathbb{Q}]$, it follows that

$$\prod_{j=1}^{d}\prod_{v\in S_j} |\rho_v(\alpha_i)\sigma_j(qu)-p_i|_v = |\alpha_i qu - p_i| \tag{2.9}$$

for all integers $i = 1, \dots, r$.

Now, for each $v \in S$, we define $d + r$ linearly independent linear forms in $d + r$ variables as follows: For $j = 1, 2, \dots, d$ and for $v \in S_j$ and for each integer i satisfying $1 \leqslant i \leqslant r$, we let

$$L_{v,i}(x_1, \dots, x_r, \dots, x_{r+d}) = x_i - \rho_v(\alpha_i)x_{j+r},$$

and when the integer i in the range $r + 1 \leqslant i \leqslant d + r$, we let

$$L_{v,i}(x_1, \dots, x_r, \dots, x_{r+d}) = x_i.$$

For each $v \in S\backslash M_\infty$ and for each integer i satisfying $1 \leqslant i \leqslant r + d$, we let

$$L_{v,i}(x_1, \dots, x_r, \dots, x_{r+d}) = x_i.$$

Let $\mathbf{X}$ be the element in K^{d+r} of the form

$$\mathbf{X} = (p_1, p_2, \dots, p_r, q\sigma_1(u), \dots, q\sigma_d(u)) \in K^{d+r}.$$

In order to apply Theorem 2.1, we need to estimate the following quantity

$$\prod_{v\in S}\prod_{j=1}^{d+r} \frac{|L_{v,j}(\mathbf{X})|_v}{\|\mathbf{X}\|_v}. \tag{2.10}$$

Using the fact that $L_{v,j}(\mathbf{X}) = q\sigma_j(u)$, for $r + 1 \leqslant j \leqslant d + r$, we obtain

$$\prod_{v\in S}\prod_{j=r+1}^{d+r} |L_{v,j}(\mathbf{X})|_v = \prod_{v\in S}\prod_{j=r+1}^{d+r} |q\sigma_j(u)|_v = \prod_{v\in S}\prod_{j=r+1}^{d+r} |q|_v \prod_{j=r+1}^{d+r}\prod_{v\in S} |\sigma_j(u)|_v.$$

Since $\sigma_j(u)$ are S-units, by the product formula, we obtain

$$\prod_{v\in S} |\sigma_j(u)|_v = \prod_{v\in M_K} |\sigma_j(u)|_v = 1.$$

Consequently, the above equality implies

$$\prod_{v\in S}\prod_{j=r+1}^{d+r} |L_{v,j}(\mathbf{X})|_v = \prod_{v\in S}\prod_{j=r+1}^{d+r} |q|_v \leqslant \prod_{v\in M_\infty}\prod_{j=r+1}^{d+r} |q|_v = \prod_{j=r+1}^{d+r} |q|^{\sum_{v\in M_\infty} d(\rho_v)/[K:\mathbb{Q}]}.$$

Then, from the formula $\sum_{v\in M_\infty} d(\rho_v) = [K:\mathbb{Q}]$, we get

$$\prod_{v\in S}\prod_{j=r+1}^{d+r} |L_{v,j}(\mathbf{X})|_v \leqslant \prod_{v\in M_\infty}\prod_{j=r+1}^{d+r} |q|_v = \prod_{j=r+1}^{d+r} |q|^{\sum_{v\in M_\infty} d(\rho_v)/[K:\mathbb{Q}]} = |q|^d. \tag{2.11}$$

Now we estimate the denominators of the product in (2.10) as follows: We have

$$\prod_{v\in S}\prod_{j=1}^{d+r} \|\mathbf{X}\|_v \geqslant \prod_{v\in M_K}\prod_{j=1}^{d+r} \|\mathbf{X}\|_v = \prod_{j=1}^{d+r}\left(\prod_{v\in M_K} \|\mathbf{X}\|_v\right) = \prod_{j=1}^{d+r} H(\mathbf{X}),$$

since $\|\mathbf{X}\|_v \leqslant 1$ for all $v \notin S$. Thus, we get

$$\prod_{v\in S}\prod_{j=1}^{d+r} \|\mathbf{X}\|_v \geqslant H(\mathbf{X})^{d+r}. \tag{2.12}$$

By (2.10), (2.11), and (2.12), it follows that

$$\prod_{v\in S}\prod_{j=1}^{d+r} \frac{|L_{v,j}(\mathbf{X})|_v}{\|\mathbf{X}\|_v} \leqslant \frac{1}{H^{d+r}(\mathbf{X})}|q|^d \prod_{i=1}^{r} |\alpha_i q u - p_i|.$$

Thus, from (2.2), we have

$$\prod_{v\in S}\prod_{j=1}^{d+r} \frac{|L_{v,j}(\mathbf{X})|_v}{||\mathbf{X}||_v} \leqslant \frac{1}{H^{d+r}(\mathbf{X})}|q|^d \frac{1}{H^{r\varepsilon}(u)}\frac{1}{|q|^{d+r\varepsilon}} = \frac{1}{H^{d+r}(\mathbf{X})}\frac{1}{(|q|H(u))^{r\varepsilon}}.$$

Notice that

$$\begin{aligned}
H(\mathbf{X}) &= \prod_{v\in M_K} \max\{|p_1|_v, \dots, |p_r|_v, |q\sigma_1(u)|_v, \dots, |q\sigma_d(u)|_v\}\\
&= \prod_{v\in S} \max\{|p_1|_v, \dots, |p_r|_v, |q\sigma_1(u)|_v, \dots, |q\sigma_d(u)|_v\}\\
&\leqslant \prod_{v\in S} \max\{|qp_1\cdots p_r|_v, \dots, |qp_1\cdots p_r|_v, |qp_1\cdots p_r\sigma_1(u)|_v, \dots, |qp_1\cdots p_r\sigma_d(u)|_v\}\\
&\leqslant |qp_1\cdots p_r| \prod_{v\in S} \max\{1, |\sigma_1(u)|_v, \dots, |\sigma_d(u)|_v\}\\
&\leqslant |qp_1\cdots p_r| \left(\prod_{v\in S} \max\{1, |\sigma_1(u)|_v\}\right)\cdots\left(\prod_{v\in S} \max\{1, |\sigma_d(u)|_v\}\right)\\
&= |qp_1\cdots p_r| H^d(u).
\end{aligned}$$

By using the inequality $||x|-|y|| \leqslant |x-y|$ and the fact, by Lemma 2.3, that $H(u)\to\infty$ when u varies over all the tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}$, by (2.2), we conclude that $|p_i| \leqslant |\alpha_i q u| + 1$. Since

$|u|^{\frac{1}{d}} \leqslant H(u)$, for all integers i satisfying $1 \leqslant i \leqslant r$, we get

$$|p_i| \leqslant |\alpha_i qu| + 1 \leqslant |q||\alpha_i|H^d(u) + 1 \leqslant |q|H^{2d}(u)$$

holds true for all but finitely many tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}$. By combining both these observations, we obtain $H(\mathbf{X}) \leqslant |q|^{r+1}H(u)^{2rd+d}$, and hence, we get $H(\mathbf{X})^{1/(d(2r+1))} \leqslant |q|H(u)$. Therefore,

$$\prod_{v\in S}\prod_{j=1}^{d+r}\frac{|L_{v,j}(\mathbf{X})|_v}{||\mathbf{X}||_v} \leqslant \frac{1}{H^{d+r}(\mathbf{X})}\frac{1}{(|q|H(u))^{r\varepsilon}} \leqslant \frac{1}{H(\mathbf{X})^{d+r+(r\varepsilon)/(2rd+1)}} = \frac{1}{H(\mathbf{X})^{d+r+\varepsilon'}},$$

for some $\varepsilon' > 0$ holds true for infinitely many tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}$. Then, by Theorem 2.1, there exists a proper subspace of K^{d+r} that contain infinitely many $\mathbf{X} \in \mathcal{B}$. That is, we have a non-trivial relation

$$a_1p_1 + a_2p_2 + \cdots + a_rp_r + b_1q\sigma_1(u) + \cdots + b_dq\sigma_d(u) = 0, \quad a_i, b_j \in K, \tag{2.13}$$

holds true for all the tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}_1$ for some infinite subset $\mathcal{B}_1$ of $\mathcal{B}$.

By the hypothesis, we know that $\alpha_i qu$ is not a pseudo-Pisot number for some integer i. Without loss of generality, we can assume that for each $(u, q, p_1, \dots, p_r) \in \mathcal{B}_1$, we have $|\alpha_r qu| > 1$ and $\alpha_r qu$ is not a pseudo-Pisot number. Under the same hypothesis as in [3, Lemma 3], Corvaja and Zannier established a relation of the form $a_1p + b_1q\sigma_1(u) + \cdots + b_dq\sigma_d(u) = 0$. Therefore, in view their work, it is enough to show the existence of a non-trivial linear relation as in (2.13) with $a_1 = a_2 = \cdots = a_{r-1} = 0$.

Claim 1. At least one of the functions of b_j is non-zero in the relation (2.13).

If possible, suppose $b_i = 0$ for all integers $i = 1, 2, \dots, r$. Then from (2.13), we get

$$a_1p_1 + a_2p_2 + \cdots + a_rp_r = 0 \quad \text{with } a_i \in K. \tag{2.14}$$

If all the functions of a_i are rational numbers, then, dividing by qu, we obtain

$$a_1\frac{p_1}{qu} + \cdots + a_r\frac{p_r}{qu} = 0 \tag{2.15}$$

holds true for all tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}_1$. For each such tuple, the inequality

$$0 < |\alpha_i qu - p_i| < \frac{1}{H^{\varepsilon}(u)|q|^{\frac{d}{r}+\epsilon}} \iff 0 < \left|\alpha_i - \frac{p_i}{qu}\right| < \frac{1}{H^{\varepsilon}(u)|u||q|^{1+\frac{d}{r}+\epsilon}}$$

holds true for every $i = 1, \dots, r$. As the tuples $(u, q, p_1, \dots, p_r)$ vary over all the elements in $\mathcal{B}_1$, by Lemma 2.3, we have $H(u) \to \infty$. Therefore, we conclude, by (2.15), that

$$a_1\alpha_1 + \cdots + a_r\alpha_r = 0.$$

Since not all functions of a_i are 0, this is a contradiction to the fact that functions of α_i are $\mathbb{Q}$-linearly independent. Hence we conclude that at least one of functions of a_i is algebraic irrational. Also, the sequence $(\frac{p_1}{qu}, \dots, \frac{p_r}{qu})$ tends to $(\alpha_1, \dots \alpha_r)$ and since none of the functions of α_i are zero, there exists an infinite subset $\mathcal{B}_2$ of $\mathcal{B}_1$ such that for any tuple $(u, q, p_1, \dots, p_r) \in \mathcal{B}_2$ satisfying $p_i \neq 0$ for all integers $i = 1, 2, \dots, r$.

Let $\{a_1, a_2, \dots, a_m\}$ be the maximal $\mathbb{Q}$-linearly independent subset of the set $\{a_1, \dots, a_r\}$, if necessary, by renaming the indices. Then we can write

$$a_{m+i} = c_{i1}a_1 + \cdots + c_{im}a_m, \quad \text{where } c_{im} \in \mathbb{Q}, \quad \text{for all } 1 \leqslant i \leqslant r - m.$$

Thus, by substituting the values of a_{m+i} in (2.14), we get

$$a_1p_1 + \cdots + a_mp_m + (c_{11}a_1 + \cdots + c_{1m}a_m)p_{m+1} \cdots + (c_{(r-m)1}a_1 + \cdots + c_{(r-m)m}a_m)p_r = 0.$$

We rewrite this equality in the following form:

$$a_1(p_1 + c_{11}p_{m+1} + \cdots + c_{(r-m)1}p_r) + \cdots + a_m(p_m + c_{1m}p_{m+1} + \cdots + c_{(r-m)m}p_r) = 0. \tag{2.16}$$

Since $a_1, \dots, a_m$ are $\mathbb{Q}$-linearly independent, by (2.16), we obtain

$$p_1 + c_{11}p_{m+1} + \cdots + c_{(r-m)1}p_r = 0 = \cdots = p_m + c_{1m}p_{m+1} + \cdots + c_{(r-m)m}p_r. \tag{2.17}$$

From (2.17), we get a relation of the form (2.14) with rational coefficients, which is again not possible as observed earlier. Thus this proves Claim 1.

Claim 2. There exists a non-trivial relation as in (2.13) with $a_i = 0$ for all $i = 1, 2, \dots, r-1$.

We first prove that there exists a relation as in (2.13) with $a_1 = 0$. If possible, we assume that $a_1 \neq 0$. Then by rewriting the relation (2.13) we obtain

$$p_1 = -\frac{a_2}{a_1}p_2 - \cdots - \frac{a_r}{a_1}p_r - \frac{b_1}{a_1}q\sigma_1(u) - \cdots - \frac{b_d}{a_1}q\sigma_d(u). \tag{2.18}$$

Case 1. $\sigma_j(\frac{b_1}{a_1}) \neq \frac{b_j}{a_1}$ for some integer j satisfying $2 \leqslant j \leqslant d$.
By applying the automorphism σ_j on both sides of (2.18), we get

$$p_1 = -\sigma_j\left(\frac{a_2}{a_1}\right)p_2 - \cdots - \sigma_j\left(\frac{a_r}{a_1}\right)p_r - \sigma_j\left(\frac{b_1}{a_1}\right)q\sigma_j\circ\sigma_1(u) - \cdots - \sigma_j\left(\frac{b_d}{a_1}\right)q\sigma_j\circ\sigma_d(u).$$

By subtracting this relation with (2.18), we get a relation involving the terms only with $p_2, \dots, p_r$, $\sigma_1(u), \dots, \sigma_d(u)$. Such a relation is non-trivial, as the coefficient of $\sigma_j(u)$ is $\sigma_j(b_1/a_1) - \frac{b_j}{a_1} \neq 0$.

Case 2. $\frac{b_j}{a_1} = \sigma_j(\frac{b_1}{a_1})$ for all integers $j = 2, 3, \dots, d$.

Note that $b_1 \neq 0$. If not, then $0 = \sigma_j(b_1/a_1) = b_j/a_1$ for every integer j and hence $b_i = 0$ for all integers i, which contradicts Claim 1. By putting $\lambda = -b_1/a_1$, we rewrite (2.18) as

$$p_1 = -\frac{a_2}{a_1}p_2 - \cdots - \frac{a_r}{a_1}p_r + q(\sigma_1(\lambda)\sigma_1(u) + \cdots + \sigma_d(\lambda)\sigma_d(u)). \tag{2.19}$$

Clearly $a_i, b_j \in K$, but it is not necessary that λ belongs to k. If $\lambda \notin k$, then there exists an automorphism $\tau \in \mathcal{H}$ with $\tau(\lambda) \neq \lambda$. By applying the automorphism τ on both sides of (2.19) and subtract with (2.19) to eliminate p_1, we obtain

$$p_2(\tau(a_2/a_1) - a_2/a_1) + \cdots + p_r(\tau(a_r/a_1) - a_r/a_1) + (\lambda - \tau(\lambda))\sigma_1(u)$$
$$+ \sum_{i=2}^{d}(\sigma_i(\lambda)\sigma_i(u) - \tau\circ\sigma_i(u)) = 0.$$

Note that $\tau\circ\sigma_j$ coincides on k with σ_i for some integer i and since $\tau \in \mathcal{H}$ and $\sigma_2, \dots, \sigma_d \notin \mathcal{H}$, none of the $\tau\circ\sigma_j$ with $j \geqslant 2$ belongs in $\mathcal{H}$. Hence the above relation can be written as a linear combination of $p_2, \dots, p_r$ and $\sigma_i(u)$ with the property that the coefficient of $\sigma_1(u)$ is $\lambda - \tau(\lambda) \neq 0$. Therefore, we obtain a non-trivial linear relation among the $p_2, \dots, p_r$ and $\sigma_i(u)$.

If $\lambda \in k$, then by adding $-\alpha_1 qu$ on both sides of the equality (2.19), we get

$$|p_1 - \alpha_1 qu| = \left|-\frac{a_2}{a_1}p_2 - \cdots - \frac{a_r}{a_1}p_r + (\lambda - \alpha_1)q\sigma_1(u) + q\sigma_2(\lambda)\sigma_2(u) + \cdots + q\sigma_d(\lambda)\sigma_d(u)\right|.$$

Then from (2.2), we get

$$\left|-\frac{a_2}{a_1}p_2 - \cdots - \frac{a_r}{a_1}p_r + (\lambda - \alpha_1)q\sigma_1(u) + q\sigma_2(\lambda)\sigma_2(u) + \cdots + q\sigma_d(\lambda)\sigma_d(u)\right| < \frac{1}{H^{\varepsilon}(u)q^{\frac{d}{r}+\varepsilon}}. \tag{2.20}$$

Then just like the inequality in (2.9), we have the following important observation:

$$\prod_{j=1}^{d}\prod_{v\in S_j}\left|-\rho_v\left(\frac{a_2}{a_1}\right)p_2 - \cdots - \rho_v\left(\frac{a_r}{a_1}\right)p_r\right.$$
$$\left. + (\rho_v(\lambda) - \rho_v(\alpha_1))q(\rho_v\circ\sigma_1)(u) + \cdots + q(\rho_v\circ\sigma_d)(\lambda)(\rho_v\circ\sigma_d)(u)\right|_v$$
$$= \left|-\frac{a_2}{a_1}p_2 - \cdots - \frac{a_r}{a_1}p_r + (\lambda - \alpha_1)q\sigma_1(u) + q\sigma_2(\lambda)\sigma_2(u) + \cdots + q\sigma_d(\lambda)\sigma_d(u)\right|.$$

For each $v \in M_\infty$ and $j = 1, \dots, d$, we define $v(j)$ such that $\rho_v\circ\sigma_j = \sigma_{v(j)}$ on the field k, where $\{v(1), \dots, v(d)\}$ is a permutation of $\{1, \dots, d\}$. Hence the above relation can be written as a linear combination of $p_2, \dots, p_r$ and $\sigma_{v(1)}, \dots, \sigma_{v(d)}$. Therefore there exist algebraic numbers

$c_{v(1)}, \dots, c_{v(d)} \in K$, not all zero, such that

$$\prod_{j=1}^{d} \prod_{v \in S_j} \left| -\rho_v\left(\frac{a_2}{a_1}\right)p_2 - \cdots - \rho_v\left(\frac{a_r}{a_1}\right)p_r \right.$$

$$\left. + (\rho_v(\lambda) - \rho_v(\alpha_1))q(\rho_v \circ \sigma_1)(u) + \cdots + q(\rho_v \circ \sigma_d)(\lambda)(\rho_v \circ \sigma_d)(u) \right|_v$$

$$= \prod_{j=1}^{d} \prod_{v \in S_j} \left| -\rho_v\left(\frac{a_2}{a_1}\right)p_2 - \cdots - \rho_v\left(\frac{a_r}{a_1}\right)p_r + c_{v(1)}q\sigma_{v(1)}(u) + \cdots + c_{v(d)}q\sigma_{v(d)}(u) \right|_v .$$

Hence, by (2.20), we have

$$\prod_{j=1}^{d} \prod_{v \in S_j} \left| -\rho_v\left(\frac{a_2}{a_1}\right)p_2 - \cdots - \rho_v\left(\frac{a_r}{a_1}\right)p_r + c_{v(1)}q\sigma_{v(1)}(u) + \cdots + c_{v(d)}q\sigma_{v(d)}(u) \right|_v < \frac{1}{H^{\epsilon}(u)q^{\frac{d}{r}+\epsilon}}. \tag{2.21}$$

Since the inequality (2.21) holds true for all the tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}_1$, we can apply Theorem 2.1 suitably.

For each $v \in M_\infty$, since $c_{v(1)}, \dots, c_{v(d)} \in K$, not all zero, we let $k_v \in \{v(1), \dots, v(d)\}$ such that $c_{k_v} \neq 0$. Now, for each $v \in S$, we define $r + d - 1$ linearly independent linear forms in $r + d - 1$ variables as follows: for each $j = 1, 2, \dots, d$ and for each $v \in S_j$, we define

$$L_{v,i}(x_1, \dots, x_r, \dots, x_{r+d-1}) = x_i - \rho_v(\alpha_i)x_{r+j-1}$$

for each $i = 1, 2, \dots, r - 1$;

$$L_{v,r-1+k_v}(x_1, \dots, x_r, \dots, x_{r+d-1})$$
$$= -\rho_v\left(\frac{a_2}{a_1}\right)x_1 - \cdots - \rho_v\left(\frac{a_r}{a_1}\right)x_{r-1} + c_{v(1)}x_{r-1+v(1)} + \cdots + c_{v(d)}x_{r-1+v(d)};$$

for each $r \leqslant m \neq r - 1 + k_v \leqslant r + d - 1$, we define

$$L_{v,m}(x_1, \dots, x_r, \dots, x_{r+d-1}) = x_m;$$

and for each $v \in S \backslash M_\infty$ and for each integer i in the range $1 \leqslant i \leqslant r + d - 1$, we consider

$$L_{v,i}(x_1, \dots, x_r, \dots, x_{r+d-1}) = x_i.$$

Since $c_{k_v} \neq 0$, it follows that for each $v \in S$, the linear forms $L_{v,1}, \dots, L_{v,r+d-1}$ are linearly independent.

Write the special points in K^{r+d-1} as

$$\mathbf{X} = (p_2, \dots, p_r, q\sigma_1(u), \dots, q\sigma_d(u)) \in K^{r+d-1}.$$

Then, by Theorem 2.1, there exists a proper subspace of K^{r+d-1} which contain infinitely many points $\mathbf{X} = (p_2, \dots, p_r, q\sigma_1(u), \dots, q\sigma_d(u))$. Hence, we get a non-trivial relation

$$a_2' p_2 + \cdots + a_r' p_r + b_1' q\sigma_1(u) + \cdots + b_d' q\sigma_d(u) = 0, \quad a_i', b_i' \in K \tag{2.22}$$

holds true for infinitely many tuples $(p_2, \dots, p_r, q\sigma_1(u), \dots, q\sigma_d(u))$. By Claim 1, we can always assume that not all functions of b_i are zero. Thus, we obtain a non-trivial relation in $r + d - 1$ tuples.

By continuing this process, inductively, we can get a non-trivial relation with $a_i = 0$ for all integers $i = 1, 2, \dots, r - 1$. That is,

$$a_r'' p_r + b_1'' q\sigma_1(u) + \cdots + b_d'' q\sigma_d(u) = 0, \quad a_r'', \ b_i'' \in K$$

holds true for infinitely many tuples $(p_r, q\sigma_1(u), \dots, q\sigma_d(u))$ where functions of b_i'' are not all zero. This proves Claim 2. We then conclude exactly as in [3, Lemma 3] to complete the proof of this lemma. □

Lemma 2.5. *Let K be a number field of degree n which is Galois over $\mathbb{Q}$ and S be a finite set of places on K which contains all the archimedean places. Let $k \subset K$ be a subfield of degree d over $\mathbb{Q}$ for some integer $d \geqslant 1$ and $\alpha_1, \dots, \alpha_d$ be any elements of K. For a given real number $\epsilon > 0$, let*

$$\mathcal{B} = \left\{ (u, q, p) \in (\mathcal{O}_S^\times \cap k) \times \mathbb{Z}^2 \, : \, 0 < |\alpha_1 q u_1 + \cdots + \alpha_d q u_d - p| < \frac{1}{H^\varepsilon(u_1)|q|^{d+\varepsilon}} \right\}, \tag{2.23}$$

where $u = u_1$ and $u_2, \dots, u_d$ are the other conjugates of u and for each triple $(u, q, p) \in \mathcal{B}$, there exists an integer $i \in \{1, \dots, d\}$ such that $|q\alpha_i u_i| > 1$ and $q\alpha_i u_i$ is not a pseudo-Pisot number. If $\mathcal{B}$ is infinite, then there exist a proper subfield $k' \subset k$, a non-zero element $u' \in k$ and an infinite subset $\mathcal{B}' \subset \mathcal{B}$ such that for all triples $(u, q, p) \in \mathcal{B}'$, we have $u/u' \in k'$.

Proof. Note that $d \geqslant 2$ because $\mathbb{Q}$ does not admit any proper subfield in it. Let $\mathcal{H} := \mathrm{Gal}(K/k) \subset \mathrm{Gal}(K/\mathbb{Q}) = \mathcal{G}$ be the subgroup of the Galois group $\mathcal{G}$ fixing k. Since K is Galois over $\mathbb{Q}$, we have that K is Galois over k and $|\mathcal{G}/\mathcal{H}| = d$. Therefore, among the n embeddings on K, there are d embeddings $\sigma_1, \dots, \sigma_d$ (with σ_1 is the identity) which are the complete set of representatives of the left cosets of $\mathcal{H}$ in $\mathcal{G}$ and more precisely, we have

$$\mathcal{G}/\mathcal{H} := \{\mathcal{H}, \sigma_2\mathcal{H}, \dots, \sigma_d\mathcal{H}\}.$$

For each $\rho \in \mathrm{Gal}(K/\mathbb{Q})$ and for any triple $(u, q, p) \in \mathcal{B}$, with the rule in (2.6), we have

$$\begin{aligned} |\alpha_1 q u_1 + \cdots + \alpha_d q u_d - p|^{d(\rho)/[K:\mathbb{Q}]} &= |\rho(\alpha_1)\rho(q u_1) + \cdots + \rho(\alpha_d)\rho(q u_d) - \rho(p)|_\rho \\ &= |\rho(\alpha_1) q \rho(u_1) + \cdots + \rho(\alpha_d) q \rho(u_d) - p|_\rho. \end{aligned} \tag{2.24}$$

For each $v \in M_\infty$, let ρ_v be an automorphism defining the valuation v, according to (2.6): $|x|_v := |x|_{\rho_v}$. Then the set $\{\rho_v \, : \, v \in M_\infty\}$ represents the left cosets of the subgroup generated by the complex conjugation in $\mathcal{G}$. For $j = 1, 2, \dots, d$, let S_j be the subset of M_∞ formed by those valuation v such that $\rho_v|_k = \sigma_j : k \to \mathbb{C}$. Note that $S_1 \cup \dots \cup S_d = M_\infty$. Thus, we have $M_\infty = \{\rho_v \, : \, v \in M_\infty\}$

and for each triple $(u, q, p) \in \mathcal{B}$, we obtain

$$\prod_{v\in M_\infty} |\rho_v(\alpha_1)q\rho_v(u_1) + \cdots + \rho_v(\alpha_d)q\rho_v(u_d) - p|_v$$

$$= \prod_{j=1}^{d} \prod_{v\in S_j} |\rho_v(\alpha_1)q\sigma_j(u_1) + \cdots + \rho_v(\alpha_d)q\sigma_j(u_d) - p|_v.$$

By (2.24), we see that

$$\prod_{v\in M_\infty} |\rho_v(\alpha_1)q\rho_v(u_1) + \cdots + \rho_v(\alpha_d)q\rho_v(u_d) - p|_v = \prod_{v\in M_\infty} |\alpha_1 qu_1 + \cdots + \alpha_d qu_d - p|^{d(\rho)/[K:\mathbb{Q}]}$$

$$= |\alpha_1 qu_1 + \cdots + \alpha_d qu_d - p|^{\sum_{v\in M_\infty} d(\rho_v)/[K:\mathbb{Q}]}.$$

Then, by the formula $\sum_{v\in M_\infty} d(\rho_v) = [K : \mathbb{Q}]$, it follows that

$$\prod_{j=1}^{d} \prod_{v\in S_j} |\rho_v(\alpha_1)q\sigma_j(u_1) + \cdots + \rho_v(\alpha_d)q\sigma_j(u_d) - p|_v = |\alpha_1 qu_1 + \cdots + \alpha_d qu_d - p|. \tag{2.25}$$

Now, for each $v \in S$, we define $d+1$ linearly independent linear forms in $d+1$ variables as follows: For each $j = 1, 2, \dots, d$ and for an archimedean place $v \in S_j$, we define

$$L_{v,0}(x_0, \dots, x_d) = -x_0 + \rho_v(\alpha_1)x_1 + \cdots + \rho_v(\alpha_d)x_d$$

and for any integer i satisfying $0 < i \leqslant d$, we define

$$L_{v,i}(x_0, \dots, x_d) = x_i.$$

Also, for any $v \in S\backslash M_\infty$ and for any integer i satisfying $0 \leqslant i \leqslant d$, we put

$$L_{v,i}(x_0, \dots, x_d) = x_i.$$

Clearly, these linear forms are $\mathbb{Q}$-linearly independent. Let the special points $\mathbf{X} \in K^{d+1}$ be of the form

$$\mathbf{X} = (p, q\sigma_1(u), \dots, q\sigma_d(u)) \in K^{d+1}.$$

In order to apply Theorem 2.1, we need to estimate the following quantity:

$$\prod_{v\in S} \prod_{j=0}^{d} \frac{|L_{v,j}(\mathbf{X})|_v}{\|\mathbf{X}\|_v}. \tag{2.26}$$

Using the fact that $L_{v,j}(\mathbf{X}) = q\sigma_j(u)$, for all $1 \leqslant j \leqslant d$ and for all v, we obtain

$$\prod_{v\in S} \prod_{j=1}^{d} |L_{v,j}(\mathbf{X})|_v = \prod_{v\in S} \prod_{j=1}^{d} |q\sigma_j(u)|_v = \prod_{v\in S} \prod_{j=1}^{d} |q|_v \prod_{j=1}^{d} \prod_{v\in S} |\sigma_j(u)|_v.$$

Since $\sigma_j(u)$ are S-units, then by the product formula we obtain

$$\prod_{v\in S}|\sigma_j(u)|_v = \prod_{v\in M_K}|\sigma_j(u)|_v = 1.$$

Consequently, from the above equality, we get

$$\prod_{v\in S}\prod_{j=1}^{d}|L_{v,j}(\mathbf{X})|_v = \prod_{v\in S}\prod_{j=1}^{d}|q|_v \leqslant \prod_{v\in M_\infty}\prod_{j=1}^{d}|q|_v = \prod_{j=1}^{d}|q|^{\sum_{v\in M_\infty} d(\rho_v)/[K:\mathbb{Q}]}.$$

Then, from the formula $\sum_{v\in M_\infty} d(\rho_v) = [K : \mathbb{Q}]$, we get

$$\prod_{v\in S}\prod_{j=1}^{d}|L_{v,j}(\mathbf{X})|_v \leqslant \prod_{v\in M_\infty}\prod_{j=1}^{d}|q|_v = \prod_{j=1}^{d}|q|^{\sum_{v\in M_\infty} d(\rho_v)/[K:\mathbb{Q}]} = |q|^d. \tag{2.27}$$

Now we estimate the product of the denominators in (2.26) as follows: Consider

$$\prod_{v\in S}\prod_{j=0}^{d}\|\mathbf{X}\|_v \geqslant \prod_{v\in M_K}\prod_{j=0}^{d}\|\mathbf{X}\|_v = \prod_{j=0}^{d}\left(\prod_{v\in M_K}\|\mathbf{X}\|_v\right),$$

since $||\mathbf{X}||_v \leqslant 1$ for all $v \notin S$. Thus, by the definition of $H(\mathbf{X})$, we conclude that

$$\prod_{v\in S}\prod_{j=0}^{d}\|\mathbf{X}\|_v \geqslant \prod_{j=0}^{d} H(\mathbf{X}). \tag{2.28}$$

By (2.25), (2.27), and (2.28), we get

$$\prod_{v\in S}\prod_{j=0}^{d}\frac{|L_{v,j}(\mathbf{X})|_v}{||\mathbf{X}||_v} \leqslant \frac{1}{H^{d+1}(\mathbf{X})}|q|^d|\alpha_1 q u_1 + \cdots + \alpha_d q u_d - p|.$$

Therefore, by (2.23), we get

$$\prod_{v\in S}\prod_{j=0}^{d}\frac{|L_{v,j}(\mathbf{X})|_v}{||\mathbf{X}||_v} \leqslant \frac{1}{H^{d+1}(\mathbf{X})}|q|^d\frac{1}{H^{\varepsilon}(u)}\frac{1}{|q|^{d+\varepsilon}} = \frac{1}{H^{d+1}(\mathbf{X})}\frac{1}{(|q|H(u))^{\varepsilon}}.$$

First note that

$$|p| \leqslant |\alpha_1 q u_1 + \cdots + \alpha_d q u_d| + 1 \leqslant |q||\alpha|\, H(u)^d\, d,$$

where $|\alpha| = \max\{|\alpha_i| : i = 1, 2, \dots, d\}$ and every conjugate of u has absolute value bounded by its Weil height power d. Hence, we get

$$|p| \leqslant C(\alpha, d)|q|H^d(u),$$

where $C(\alpha, d)$ is a positive constant depends only on α and d. Since $H(\mathbf{X}) \leqslant |q||p|H(u)^d$, we get

$$H(\mathbf{X}) \leqslant C'|q|^2 H(u)^{2d} \leqslant C'(|q|H(u))^{2d} \Longrightarrow |q|H(u) \geqslant C'' H(\mathbf{X})^{1/(2d)}.$$

where C' and C'' are positive constants that depend only on α and d and hence the last inequality becomes

$$\prod_{v\in S}\prod_{j=0}^{d} \frac{|L_{v,j}(\mathbf{X})|_v}{||\mathbf{X}||_v} \leqslant \frac{1}{H(\mathbf{X})^{d+1+\varepsilon'}},$$

for some $\varepsilon' > 0$ which holds for infinitely many points $\mathbf{X}$. Therefore, by Theorem 2.1, there exists a proper subspace of K^{d+1} which contain infinitely many points $\mathbf{X} = (p, q\sigma_1(u), \dots, q\sigma_d(u))$. It means that we obtain a non-trivial relation

$$a_0 p + a_1 q\sigma_1(u) + \cdots + a_d q\sigma_d(u) = 0, \quad a_i \in K, \tag{2.29}$$

satisfied by all the triples $(u, q, p) \in \mathcal{B}_1 \subset \mathcal{B}$ for some infinite subset $\mathcal{B}_1$ of $\mathcal{B}$. Also, for each triple $(u, q, p) \in \mathcal{B}_1$, without loss of generality, we can assume that $|q\alpha_1 u| > 1$ and $q\alpha_1 u$ is not a pseudo-Pisot number.

Since not all functions of a_i are 0, clearly, we can conclude that at least one among $a_1, \dots, a_d$ is non-zero. Now, we have the following claim.

Claim 1. There exists a non-trivial relation as in (2.29) with $a_0 = 0$.

Suppose that $a_0 \neq 0$. Then we rewrite the relation (2.29) as

$$p = -\frac{a_1}{a_0} q\sigma_1(u) - \cdots - \frac{a_d}{a_0} q\sigma_d(u). \tag{2.30}$$

By considering the case when $\sigma_j(a_1/a_0) \neq a_j/a_0$, for some index $j \in \{2, \dots, d\}$, or the case when $a_j/a_0 = \sigma_j(a_1/a_0)$ for all j, we can conclude that all the coefficients a_j/a_0 are non-zero. Former case can be handled as in Case 1 in Claim 2 of Lemma 2.4. We deal with the latter case.

Put $\lambda = -a_1/a_0$. With these notations, we can re-write (2.30) as follows:

$$p = q(\sigma_1(\lambda)\sigma_1(u) + \cdots + \sigma_d(\lambda)\sigma_d(u)).$$

In the proof of Claim 2 of Lemma 2.4, we had the two possibilities, namely, either $\lambda \in k$ or $\lambda \notin k$. Here also, the proof for the case $\lambda \notin k$ is similar to that of the proof of Claim 2 of Lemma 2.4. Therefore we consider the case $\lambda \in k$. By adding $-\alpha_1 q\sigma_1(u) - \cdots - \alpha_d q\sigma_d(u)$ to both sides in the above equality, we get

$$\begin{aligned} |p - (\alpha_1 q\sigma_1(u) + \cdots + \alpha_d q\sigma_d(u))| &= |p - (\alpha_1 q u_1 + \cdots + \alpha_d q u_d)| \\ &= |(\lambda - \alpha_1) q\sigma_1(u) + (\sigma_2(\lambda) - \alpha_2) q\sigma_2(u) + \cdots + (\sigma_d(\lambda) - \alpha_d) q\sigma_d(u)|. \end{aligned}$$

Therefore by (2.23), we get

$$0 < |(\lambda - \alpha_1) q\sigma_1(u) + (\sigma_2(\lambda) - \alpha_2) q\sigma_2(u) + \cdots + (\sigma_d(\lambda) - \alpha_d) q\sigma_d(u)| < \frac{1}{|q|^{d+\varepsilon}} \frac{1}{H^{\varepsilon}(u)}.$$

Then dividing by q on both sides to get

$$0 < |(\lambda - \alpha_1)\sigma_1(u) + (\sigma_2(\lambda) - \alpha_2)\sigma_2(u) + \cdots + (\sigma_d(\lambda) - \alpha_d)\sigma_d(u)| < \frac{1}{|q|^{d+1+\varepsilon}} \frac{1}{H^{\varepsilon}(u)} \leqslant \frac{1}{|q|^{1+\varepsilon}} \frac{1}{H^{\varepsilon}(u)}. \tag{2.31}$$

By putting $\sigma_i(\lambda) - \alpha_i = \beta_i$ for all integers i, we note that not all functions of β_i are zero. Then by re-writing (2.31), we get

$$|\beta_1\sigma_1(u) + \cdots + \beta_d\sigma_d(u)| < \frac{1}{|q|^{1+\varepsilon}} \frac{1}{H^{\varepsilon}(u)} \tag{2.32}$$

holds true for all triples $(u, q, p) \in \mathcal{B}_1$. In order to apply Lemma 2.1, we distinguish two cases, namely, $\beta_1 = 0$ and $\beta_1 \neq 0$ as follows.

Suppose $\beta_1 = 0$. In this case, $\sigma_1(\lambda) = \alpha_1$ and hence the algebraic number $q\alpha_1 u = q\lambda u$. Since $\alpha_1 qu$ is not a pseudo-Pisot number, we get that $q\lambda u$ is not a pseudo-Pisot number. Therefore,

$$\max\{|\sigma_2(q\lambda u)|, \ldots, |\sigma_d(q\lambda u)|\} \geqslant 1.$$

This implies that

$$\max\{|\sigma_2(u)|, \ldots, |\sigma_d(u)|\} \geqslant \frac{1}{|q|} \max\{|\sigma_2(\lambda)|, \ldots, |\sigma_d(\lambda)|\}^{-1}. \tag{2.33}$$

Since not all functions of β_i are zero, let $\beta_{i_1}, \ldots, \beta_{i_r}$ be non-zero elements among $\beta_2, \ldots, \beta_d$ and by (2.32), we get

$$|\beta_{i_1}\sigma_{i_1}(u) + \cdots + \beta_{i_r}\sigma_{i_r}(u)| < \frac{1}{|q|^{1+\varepsilon}} \frac{1}{H^{\varepsilon}(u)}. \tag{2.34}$$

holds true for all triples $(u, q, p) \in \mathcal{B}_1$. Thus by (2.33) and (2.34), for all triples $(u, q, p) \in \mathcal{B}_1$, the inequality

$$|\beta_{i_1}\sigma_{i_1}(u) + \cdots + \beta_{i_r}\sigma_{i_r}(u)| < \max\{|\sigma_1(u)|, \ldots, |\sigma_d(u)|\}H^{-\varepsilon}(u)$$

holds true. Therefore by Lemma 2.1, with the distinguished place ω corresponding to the identity embedding, $n = i_r$ and $\lambda_{i_j} = \beta_{i_j}$ for $1 \leqslant j \leqslant r$, we get an infinite subset $\mathcal{B}_2 \subset \mathcal{B}_1$ such that for all triples $(u, q, p) \in \mathcal{B}_2$ there exists a non-trivial relation of the form

$$s_1 q\sigma_1(u) + \cdots + s_d q\sigma_d(u) = 0$$

holds true for some $s_1, \ldots, s_d \in K$. Therefore, by Lemma 2.2, there exist an infinite subset $\mathcal{B}_3$ of $\mathcal{B}_2$ and a non-trivial relation of the form $a\sigma_j(u) + b\sigma_i(u) = 0$ for some distinct integers i and j and $a, b \in K^{\times}$ satisfied by all the triples $(u, q, p) \in \mathcal{B}_3$. Hence,

$$-\sigma_i^{-1}\left(\frac{a}{b}\right)(\sigma_i^{-1} \circ \sigma_j)(u) = u$$

is true for all triples $(u, q, p) \in \mathcal{B}_3$. Therefore, for any two triples $(u', q', p'), (u'', q'', p'') \in \mathcal{B}_3$, we have

$$\sigma_i^{-1} \circ \sigma_j(u'/u'') = u'/u''.$$

That is, the element u'/u'' is fixed by the automorphism $\sigma_i^{-1} \circ \sigma_j \notin \mathcal{H}$, and hence u'/u'' belongs to the proper subfield k' of k which is fixed by the subgroup generated by $\mathcal{H}$ and $\sigma_i^{-1} \circ \sigma_j$. To finish the proof of this lemma, fix a non-zero $u' \in k$ with $(u', q, p) \in \mathcal{B}_3$ and take any other triple $(u, q, p) \in \mathcal{B}_3$, then we can get $u/u' \in k'$.

Now we assume that $\beta_1 \neq 0$. In this case, the term $\beta_1\sigma_1(u)$ does appear in (2.32). Since $|\alpha_1 q u_1| = |\alpha_1 q u| > 1$, we see that

$$\max\{|u_1|, \dots, |u_d|\} = \{|\sigma_1(u)|, \dots, |\sigma_d(u)|\} \geqslant |u| > |\alpha_1|^{-1}|q|^{-1}$$

holds true for all pairs (u, q) where the triples (u, q, p) satisfying (2.32). Thus by (2.32), we deduce that

$$0 < |\beta_1\sigma_1(u) + \cdots + \beta_d\sigma_d(u)| < \frac{1}{|q|^{1+\varepsilon}} \frac{1}{H^{\varepsilon}(u)} < \frac{|\alpha_1| \max\{|\sigma_1(u)|, \dots, |\sigma_d(u)|\}}{|q|^{\varepsilon}H^{\varepsilon}(u)}.$$

By applying Lemma 2.1 with the distinguished place ω as in the case $\beta_1 = 0$ and with the inputs $n = d$, $\lambda_i = \beta_i$ for each integer $i = 1, \dots, d$, we conclude the same as in the case when $\beta_1 = 0$. This completes the proof of the lemma. □

3 | PROOF OF THEOREMS 1.1 AND 1.2 AND COROLLARY 1.1

Proof of Theorem 1.1. Since Γ is finitely generated multiplicative subgroup of $\overline{\mathbb{Q}}^{\times}$, by enlarging Γ, if necessary, we can reduce to the situation where $\Gamma \subset \overline{\mathbb{Q}}^{\times}$ is a group of S-units, namely,

$$\Gamma = \mathcal{O}_S^{\times} = \left\{ u \in K : \prod_{v \in S} |u|_v = 1 \right\}$$

of a suitable number field K which is Galois over $\mathbb{Q}$, with $\alpha_1, \dots, \alpha_r$ in K and for a suitable finite set of places S of K which contains all the archimedean places. Also, note that S is stable under Galois conjugation. □

Suppose that the assertion is not true. That is, the subset $\mathcal{B}$ (which is defined in (1.1)) is an infinite set. Then by inductively, we construct sequences $\{\alpha_i^{(1)}\}_{i\geqslant 0}, \dots, \{\alpha_i^{(r)}\}_{i\geqslant 0}$, whose elements are in K, with the property that for any integer $n \geqslant 0$, the numbers $(\alpha_0^{(1)} \cdots \alpha_n^{(1)})$, ..., $(\alpha_0^{(r)} \cdots \alpha_n^{(r)})$ are $\mathbb{Q}$-linearly independent, an infinite decreasing chain $\mathcal{B}_i$ of an infinite subset of $\mathcal{B}$ and an infinite strictly decreasing chain k_i of subfields of K satisfying the following:

For each integer $n \geqslant 0$, $\mathcal{B}_n \subset (k_n \times \mathbb{Z}^{r+1}) \cap \mathcal{B}_{n-1}$, $k_n \subset k_{n-1}$, $k_n \neq k_{n-1}$ and all but finitely many tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}_n$ satisfying the inequalities: $|\alpha_0^{(i)} \cdots \alpha_n^{(i)} qu| > 1$, $\alpha_0^{(i)} \cdots \alpha_n^{(i)} qu$ is not a pseudo-Pisot number for some integer $i \in \{1, \dots, r\}$, and

$$|\alpha_0^{(j)} \cdots \alpha_n^{(j)} qu - p_j| < \frac{1}{H^{\varepsilon/(n+1)}(u)|q|^{\frac{d}{r}+\varepsilon}} \text{ for each integer } j = 1, 2, \dots, r. \tag{3.1}$$

If such sequences exist, then we eventually get a contradiction to the fact that the number field K does not admit an infinite strictly decreasing chain of subfields. Therefore in order to finish the proof of the theorem, it suffices to construct such sequences.

We proceed our construction by applying induction on n: for $n = 0$, put $\alpha_0^{(j)} = \alpha_j$ for $1 \leqslant j \leqslant r$, $k_0 = K$ and $\mathcal{B}_0 = \mathcal{B}$, and we are done in this case because of our supposition.

By the induction hypothesis, we assume that $\alpha_n^{(j)}$, k_n, and $\mathcal{B}_n$ for an integer $n \geqslant 0$ exist with the property that $(\alpha_0^{(1)} \cdots \alpha_n^{(1)}), \dots, (\alpha_0^{(r)} \cdots \alpha_n^{(r)})$ are $\mathbb{Q}$-linearly independent and satisfying (3.1). Now we prove $n+1$th stage.

For each integer $j = 1, 2, \dots, r$, we let

$$\delta_j = \alpha_0^{(j)} \cdots \alpha_n^{(j)}.$$

By the induction hypothesis, the numbers $\delta_1, \dots, \delta_r$ are $\mathbb{Q}$-linearly independent and satisfy (3.1). Then by applying Lemma 2.4 with $\delta_1, \dots, \delta_r, k = k_n$, we obtain an element $\gamma_{n+1} \in k_n$, a proper subfield k_{n+1} of k_n and an infinite set $\mathcal{B}_{n+1} \subset \mathcal{B}_n$ such that all tuples $(u, q, p_1, \dots, p_r) \in \mathcal{B}_{n+1}$ satisfy $u = \gamma_{n+1} v$ with $v \in k_{n+1}$ and $\gamma_{n+1} \in k_n$. Note that since $u \in \mathcal{O}_S^{\times}$, we observe that $v \in \mathcal{O}_S^{\times}$. Hence, as u varies, we see that v also varies over $\mathcal{O}_S^{\times}$. Thus, we can assume that $(u, q, p_1, \dots, p_r) \in \mathcal{B}_{n+1}$ if and only if $(v, q, p_1, \dots, p_r) \in \mathcal{B}_{n+1}$.

Set $\alpha_{n+1}^{(j)} = \gamma_{n+1}$ for all $1 \leqslant j \leqslant r$. Clearly,

$$\alpha_0^{(j)} \cdots \alpha_n^{(j)} \alpha_{n+1}^{(j)} = \delta_j \gamma_{n+1} := \delta_{n+1}^{(j)} \text{ for all } 1 \leqslant j \leqslant r.$$

Therefore, by induction hypothesis, it is clear that $\delta_{n+1}^{(1)}, \dots, \delta_{n+1}^{(r)}$ are $\mathbb{Q}$-linearly independent. Also, by induction hypothesis, we know that for every tuple $(u, q, p_1, \dots, p_r) \in \mathcal{B}_{n+1}$, there exists an integer $i \in \{1, \dots, r\}$ satisfying $|\delta_i q u| > 1$ and $\delta_i q u$ is not a pseudo-Pisot number. Since $\delta_j q u = \delta_j q \gamma_{n+1} v = \delta_{n+1}^{(j)} q v$, for every tuple $(v, q, p_1, \dots, p_r) \in \mathcal{B}_{n+1}$, there exists an integer i such that $|\delta_{n+1}^{(i)} q v| > 1$ and $\delta_{n+1}^{(i)} q v$ is not a pseudo-Pisot number and

$$|\delta_{n+1}^{(j)} q v - p_j| = |\delta_j \gamma_{n+1} q v - p_j| = |\delta_j q u - p_j| < \frac{1}{H(\gamma_{n+1} v)^{\varepsilon/(n+1)} |q|^{\frac{d}{r}+\varepsilon}}.$$

Since $v \in k_{n+1}$, we see that

$$H(\gamma_{n+1} v) \geqslant H(\gamma_{n+1})^{-1} H(v),$$

and hence, in particular, for almost all $v \in k_{n+1}$, we get $H(\delta_{n+1} v) \geqslant H(v)^{(n+1)/(n+2)}$. Therefore, for all but finitely many tuples $(v, q, p_1, \dots, p_r) \in \mathcal{B}_{n+1}$ and for all $1 \leqslant j \leqslant r$, we have the following inequality

$$|\delta_{n+1}^{(j)} q v - p_j| < \frac{1}{H(v)^{\varepsilon/(n+2)} |q|^{\frac{d}{r}+\varepsilon}}.$$

This proves the induction and hence the theorem. □

Proof of Theorem 1.2. The proof of this theorem is similar to the proof of Theorem 1.1.

Suppose that there are infinitely many triples $(u, q, p) \in \mathcal{O}_S^\times \times \mathbb{Z}^2$ satisfying the following inequality:

$$0 < |\alpha_1 q\sigma_1(u) + \cdots + \alpha_d q\sigma_d(u) - p| \leqslant |q|^{-d-\varepsilon} H^{-\varepsilon}(u),$$

where functions of σ_i are all the embeddings of $\mathbb{Q}(u)$ to $\mathbb{C}$. Then by inductively, we construct sequences $\{\alpha_{i,0}\}_{i=0}^{\infty}, \ldots, \{\alpha_{i,d}\}_{i=0}^{\infty}$ whose elements are in K, an infinite decreasing chain $\mathcal{B}_i$ of an infinite subset of $\mathcal{B}$ and an infinite strictly decreasing chain k_i of subfields of K satisfying the following properties.

For each integer $n \geqslant 0$, $\mathcal{B}_n \subset (k_n \times \mathbb{Z}^2) \cap \mathcal{B}_{n-1}$, $k_n \subset k_{n-1}$, $k_n \neq k_{n-1}$ and all but finitely many triples $(u, q, p) \in \mathcal{B}_n$ satisfying the inequalities: $|\alpha_{i,0} \cdots \alpha_{i,n} q u_i| > 1$, $\alpha_{i,0} \cdots \alpha_{i,n} q u_i$ is not a pseudo-Pisot number for some integer $i \in \{1, \ldots, d\}$ and

$$|\alpha_{1,0} \cdots \alpha_{1,n} q\sigma_1(u) + \cdots + \alpha_{d,0} \cdots \alpha_{d,n} q\sigma_d(u) - p| < \frac{1}{H^{\varepsilon/(n+1)}(u)|q|^{d+\varepsilon}}. \tag{3.2}$$

If such sequences exist, then we eventually get a contradiction to the fact that the number field K does not admit any infinite strictly decreasing chain of subfields. Therefore in order to finish the proof of the theorem, it suffices to construct such sequences.

We proceed our construction by applying induction on n: for $n = 0$, put $\alpha_{i,0} = \alpha_i$ for each integer $i = 1, \ldots, d$, $k_0 = K$ and $\mathcal{B}_0 = \mathcal{B}$, and we are done in this case because of our supposition. By the induction hypothesis, we assume that $\alpha_{i,n}$, k_n, and $\mathcal{B}_n$ for an integer $n \geqslant 0$ and (3.2) holds true. Then by applying Lemma 2.5 with $k = k_n$ and

$$\delta_1 = \alpha_{1,0} \cdots \alpha_{1,n}, \ldots, \delta_d = \alpha_{d,0} \cdots \alpha_{d,n},$$

we obtain an element $\gamma_{n+1} \in k_n$, a proper subfield k_{n+1} of k_n and an infinite set $\mathcal{B}_{n+1} \subset \mathcal{B}_n$ such that all triples $(u, q, p) \in \mathcal{B}_{n+1}$ satisfy $u = \gamma_{n+1} v$ with $v \in k_{n+1}$. Note that since $u \in \mathcal{O}_S^\times$, we observe that $v \in \mathcal{O}_S^\times$. Hence, as u varies, we see that v varies over $\mathcal{O}_S^\times$. Thus, we can assume that $(u, q, p) \in \mathcal{B}_{n+1}$ if and only if $(v, q, p) \in \mathcal{B}_{n+1}$.

Set $\alpha_{j,n+1} = \sigma_j(\gamma_{n+1})$ for all $1 \leqslant j \leqslant d$. Clearly,

$$\alpha_{j,0} \cdots \alpha_{j,n} \alpha_{j,n+1} = \delta_j \sigma_j(\gamma_{n+1}) := \delta_{n+1}^{(j)} \quad \text{for all } 1 \leqslant j \leqslant d.$$

By the induction hypothesis, we know that for every triple $(u, q, p) \in \mathcal{B}_{n+1}$, there exists an integer i satisfying $|\delta_i q u| > 1$ and $\delta_i q u$ is not a pseudo-Pisot number. Since $\delta_j q u_j = \delta_j q \sigma_j(\gamma_{n+1} v) = \delta_{n+1}^{(j)} q \sigma_j(v)$, for every triple $(u, q, p) \in \mathcal{B}_{n+1}$, there exists an integer i such that $|\delta_{n+1}^{(i)} q u_i| > 1$ and $\delta_{n+1}^{(i)} q u_i$ is not a pseudo-Pisot number and

$$\begin{aligned}
&|\delta_{n+1}^{(1)} q\sigma_1(v) + \cdots + \delta_{n+1}^{(i)} q\sigma_i(v) + \cdots + \delta_{n+1}^{(d)} q\sigma_d(v) - p| \\
&\quad = |\delta_1 q\sigma_1(\gamma_{n+1} v) + \cdots + \delta_d q\sigma_d(\gamma_{n+1} v) - p| \\
&\quad = |\delta_1 q u_1 + \cdots + \delta_d q u_d - p| < \frac{1}{H^{\varepsilon/(n+1)}(\gamma_{n+1} v)|q|^{d+\varepsilon}}.
\end{aligned}$$

Since $v \in K$, we see that

$$H(\gamma_{n+1}v) \geqslant H(\gamma_{n+1})^{-1}H(v),$$

and hence, in particular, for almost all $v \in K$, we have $H(\delta_{n+1}v) \geqslant H(v)^{(n+1)/(n+2)}$. Therefore, for all but finitely many such triple $(v, q, p) \in \mathcal{B}_{n+1}$, we have the following inequality:

$$|\delta_{n+1}^{(1)}qv + \cdots + \delta_{n+1}^{(i)}q\sigma_i(v) + \cdots + \delta_{n+1}^{(d)}q\sigma_d(v) - p| < \frac{1}{H^{\varepsilon/(n+2)}(v)|q|^{d+\varepsilon}}$$

holds true. This proves the induction step and hence the theorem. □

Proof of Corollary 1.1. Suppose that the assertion of Corollary 1.1 is false. Then $\alpha_1, \ldots, \alpha_r$ are algebraic numbers. By choosing $\varepsilon < \eta \log |\alpha| / \log H(\alpha)$, we see that there are infinitely many tuples $(n, q, p_1, \ldots, p_r)$ in $\mathbb{Z}_{>0}^2 \times \mathbb{Z}^r$ satisfying

$$0 < |\alpha_j \alpha^n q - p_j| < \frac{1}{H(\alpha^n)^\varepsilon q^{\frac{d}{r}+\varepsilon}} \quad \text{for all } 1 \leqslant j \leqslant r.$$

Then by taking $\Gamma = \langle \alpha \rangle$ and $u = \alpha^n$ in Theorem 1.1, we get, for infinitely many values of n, $\alpha_i q \alpha^n$ is a pseudo-Pisot number for all $1 \leqslant i \leqslant r$, and, in particular, all their other conjugates have modulus less than 1.

Let K be the Galois closure of the number field $\mathbb{Q}(\alpha, \alpha_1, \ldots, \alpha_r)$ over $\mathbb{Q}$. By our assumption on α, we know that α has a conjugate β with $|\beta| > |\alpha|$. Therefore there exists an automorphism $\sigma : K \to K$ maps α to β. Hence, for all $n \in \mathbb{N}$, we have $\sigma(q\alpha_i\alpha^n) = q\sigma(\alpha_i)\beta^n$. Since $\alpha_i q \alpha^n$ is a pseudo-Pisot number for infinitely many values of n, we see that all the other conjugates of $\alpha_i q \alpha^n$ have modulus < 1. In particular, the same is true for $\sigma(q\alpha_i\alpha^n)$. But, since $|\sigma(q\alpha_i\alpha^n)| = |q\sigma(\alpha_i)||\beta|^n$, and $|\beta| > |\alpha| > 1$, this is impossible. This proves the corollary. □

ACKNOWLEDGEMENTS

We are grateful to the referee for many constructive suggestions and pointing out many typos to make the article readable and clear. Also, we are thankful to NISER, Bhubaneswar, where, in December, 2019, this work was blossomed. The first author is thankful to Professor P. Philippon for his useful discussion in the preliminary version of this article and encouragements in this work. The second author is thankful to SERB-METRICS project.

JOURNAL INFORMATION

Mathematika is owned by University College London and published by the London Mathematical Society. All surplus income from the publication of *Mathematika* is returned to mathematicians and mathematics research via the Society's research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES

1. E. Bombieri and W. Gubler, *Heights in Diophantine geometry*, New Mathematical Monographs, vol. 4, Cambridge University Press, Cambridge, 2006.
2. P. Corvaja and J. Hančl, *A transcendence criterion for infinite products*, Atti Accad. Naz. Lincei Rend. Lincei Math. Appl. **18** (2007), no. 3, 295–303.

3. P. Corvaja and U. Zannier, *On the rational approximation to the powers of an algebraic number: solution of two problems of Mahler and Mendes France*, Acta Math. **193** (2004), 175–191.
4. J. Hančl, O. Kolouch, S. Pulcerová, and J. Štěpnička, *A note on the transcendence of infinite products*, Czech. Math. J. **62** (2012), no. 3, 613–623.
5. A. Kulkarni, N. M. Mavraki, and K. D. Nguyen, *Algebraic approximations to linear combinations of powers: an extension of results by Mahler and Corvaja-Zannier*, Trans. Amer. Math. Soc. **371** (2019), 3787–3804.
6. D. Ridout, *The p-adic generalization of the Thue-Siegel-Roth theorem*, Mathematika **5** (1958), 122–124.
7. K. Roth, *Rational approximations to algebraic numbers*, Mathematika **2** (1955), no. 1, 1–20.
8. N. Saradha and R. Thangadurai, *Pillars of transcendental number theory*, Springer, Singapore, 2020.
9. W. M. Schmidt, *Diophantine approximations and diophantine equations*, Lecture Notes in Mathematics, vol. 1467, Springer, Berlin, 1991.
10. U. Zannier, *Some applications of diophantine approximation to diophantine equations (with special emphasis on the Schmidt Subspace Theorem)*, Forum, Udine, 2003.